Lior Silberman's Math 223: Problem Set 3 (due 2/2/2022)

Practice problems (recommended, but do not submit)

Section 1.6, Problems 1 (except (g)), 2-5, 7,8, 11,12, 22*, 24*.

M1. (§1.6 E8)Let $W = \left\{\underline{x} \in \mathbb{R}^5 \mid \sum_{i=1}^{5} x_i = 0\right\}$ be the set of vectors in $\mathbb{R}^5$ whose co-ordinates sum to zero. It is a subspace (but you don't have to check this). The following 8 vectors span W (you don't have to check that either). Find a subset of them which forms a basis for W. $\underline{u}_1 = (2,-3,4,-5,2)$, $\underline{u}_2 = (-6,9,-12,15,-6)$, $\underline{u}_3 = (3,-2,7,-9,1)$, $\underline{u}_4 = (2,-8,2,-2,6)$, $\underline{u}_5 = (-1,1,2,1,-3)$, $\underline{u}_6 = (0,-3,-18,9,12)$, $\underline{u}_7 = (1,0,-2,3,-2)$, $\underline{u}_8 = (2,-1,1,-9,7)$.

M2. Find a basis for the subspace $\left\{\underline{x} \in \mathbb{R}^4 \mid x_1 + 3x_2 - x_3 = 0\right\}$ of $\mathbb{R}^4$. What is the dimension?

Basis and dimension

1. Recall the space $M_n(\mathbb{R})$ of $n \times n$ matrices: each element is a square $n \times n$ array of real numbers, with addition and scalar multiplication entrywise. For $A \in M_n(\mathbb{R})$ define its *transpose* A^T by reflecting along the main diagonal: $\left(A^T\right)_{ij} = A_{ji}$. For example $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$. Call a matrix A *symmetric* if $A^T = A$ (for example $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ is symmetric but $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ isn't). The set of symmetric matrices $S \subset M_n(\mathbb{R})$ is a subspace (we'll prove this later). Find a basis for this space and compute its dimension.

*2. Let $\mathbb{R}(x)$ be the space of functions of the form $\frac{f}{g}$ where $f,g \in \mathbb{R}[x]$ are polynomials such that $g \neq 0$. $\mathbb{R}(x)$ is called "the field of rational functions in one variable, and has the same relation to the ring of polynomials $\mathbb{R}[x]$ that the rational numbers $\mathbb{Q}$ have to the ring of integers $\mathbb{Z}$. We will consider $\mathbb{R}(x)$ as a real vector space.
   (a) Show that $\frac{1}{1-x} \in \mathbb{R}(x)$ is linearly independent of the set $\left\{x^k\right\}_{k=0}^{\infty} \subset \mathbb{R}(x)$.
   RMK It's true that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ holds on the interval $(-1,1)$, but don't forget that the summation symbol on the left *does not stand* for repeated addition. Rather, it stands for a kind of limit.
   (b) Show that the subset $\left\{\frac{1}{x-a}\right\}_{a \in \mathbb{R}} \subset \mathbb{R}(x)$ is linearly independent.
   RMK The vector space $\mathbb{R}[x]$ has countable dimension, but by part (b) the dimension of $\mathbb{R}(x)$ as a real vector space is at least the cardinality of the continuum. In fact there is equality, because the cardinality of all of $\mathbb{R}(x)$ is that of the continuum.

Linear Functionals

Fix a vector space V. A *linear functional* on V is a map $\varphi\colon V \to \mathbb{R}$ such that for all $a,b \in \mathbb{R}$ and $\underline{u},\underline{v} \in V$, $\varphi(a\underline{v} + b\underline{u}) = a\varphi(\underline{v}) + b\varphi(\underline{u})$. Let $V^* \stackrel{\text{def}}{=} \{\varphi\colon V \to \mathbb{R} \mid \varphi \text{ is a linear functional}\}$ be the set of linear functionals on V (V^* is the vector space *dual* to V, in short the *dual space*).

3. (The basic examples)
   (a) Show that $\varphi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = 5x - y - 4z$ defines a linear functional on $\mathbb{R}^3$.

(b) Let φ be a linear functional on $\mathbb{R}^2$. Show that $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\cdot\varphi\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + y\cdot\varphi\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, and conclude that every linear functional on $\mathbb{R}^2$ is of the form $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = ax+by$ for some $a,b\in\mathbb{R}$.

SUPP Construct an identification of $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$, generalizing part (b).

(c) Fix a set X and a point $x\in X$. Define $e_x\colon \mathbb{R}^X\to\mathbb{R}$ by $e_x(f)=f(x)$ (this is called the "evaluation map"). Show that e_x is a linear functional.

4. Show that V^* is a subspace of $\mathbb{R}^V$, hence a vector space.

A Linear Transformation

In this problem our notation follows conventions from physics. Thus v will be a numerical parameter rather than a vector, and we write the coordinates of a vector in $\mathbb{R}^2$ as $\begin{pmatrix} x \\ t \end{pmatrix}$ rather than $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

5. In the course of his researches on electromagnetism, Henri Poincaré wrote down the following map $L_v\colon \mathbb{R}^2\to\mathbb{R}^2$ which he called the "Lorentz transformation":

$$L_v\begin{pmatrix} x \\ t \end{pmatrix} \stackrel{\text{def}}{=} \gamma_v\cdot\begin{pmatrix} x-vt \\ t-vx \end{pmatrix}.$$

Here v is a real parameter such that $|v|<1$ and γ_v is also a number, defined by $\gamma_v=\left(1-v^2\right)^{-1/2}$.

(a) Suppose $v=0.6$ so that $\gamma_v=(1-0.6^2)^{-1/2}=1.25$. Calculate $L_v\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $L_v\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $L_v\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Check that $L_v\begin{pmatrix} 2 \\ 3 \end{pmatrix} = L_v\begin{pmatrix} 3 \\ 2 \end{pmatrix} + L_v\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) Show that L_v is a linear transformation.

(c) ("Relativistic addition of velocities") Let $v,v'\in(-1,1)$ be two parameters. Show that $L_v\circ L_{v'}=L_u$ for $u=\frac{v+v'}{1+vv'}$. It is a fact that if $v,v'\in(-1,1)$ then $\frac{v+v'}{1+vv'}\in(-1,1)$ as well.
Hint: Start by showing $\gamma_v\gamma_{v'}=\frac{\gamma_u}{1+vv'}$.

RMK If $g\colon A\to B$ and $f\colon B\to C$ are functions then $f\circ g$ denotes their *composition*, the function $f\circ g\colon A\to C$ such that $(f\circ g)(a)=f(g(a))$ for all $a\in A$.

Extra credit

C1. Let V be a vector space and let $W_1,W_2\subset V$ be finite-dimensional subspaces.

(a) Show that $\dim(W_1+W_2)\le\dim W_1+\dim W_2$.

(**b) Show that $\dim(W_1+W_2)+\dim(W_1\cap W_2)=\dim W_1+\dim W_2$.

Hint Let A,B be finite sets. Then the "inclusion-exclusion" formula states $\#A+\#B=\#(A\cup B)+\#(A\cap B)$

C2. For a vector space V and a set X endow V^X with the structure of a vector space (check the axioms!). When U,V are vector space show that the set of linear maps $\mathrm{Hom}_{\mathbb{R}}(U,V)=\{f\colon U\to V\mid f\text{ is linear}\}\subset V^U$ is a subspace.

C3. (a) Let V be a vector space of dimension r over the finite field $\mathbb{F}_p$. Show that $\#V=p^r$.

(b) Combine problems C2,C3 from PS1 and part (a) to show that every finite field has p^r elements for some prime p and positive integer r.

Supplementary problems

A. Let V be a vector space and let $\varphi \in V^*$ be non-zero.
   (a) Show that $\operatorname{Ker}\varphi \stackrel{\text{def}}{=} \{\underline{v} \in V \mid \varphi(\underline{v}) = 0\}$ is a subspace.
   (*b) Show that there is $\underline{v} \in V$ satisfying $\varphi(\underline{v}) = 1$.
   (**c) Let B be a basis of $\operatorname{Ker}\varphi$, and let $\underline{v} \in V$ be as in part (b). Show that $B \cup \{\underline{v}\}$ is a basis of V.
   RMK If V is finite-dimensional this shows: $\dim V = \dim \operatorname{Ker}\varphi + 1$. In general we say that $\operatorname{Ker}\varphi$ is of *codimension* 1.

B. Let V be a vector space, W a subspace. Let $B \subset W$ be a basis for W and let $C \subset V$ be disjoint from B and such that $B \cup C$ is a basis for V (that is, we extend B until we get a basis for V).
   (a) Show that $\{\underline{v} + W\}_{\underline{v} \in C}$ is a basis for the quotient vector space V/W (V/W is defined in the supplement to PS2).
   (b) Show that $\dim W + \dim(V/W) = \dim V$.

The following problem requires some background in set theory.

C. Let V be a vector space, and let B, C be a bases of V.
   (a) Suppose one of B, C is finite, Show that the other is finite and that they have the same size.
   – We may therefore assume both B, C are infinite.
   (b) For a finite subset $A \subset B$ show that $C \cap \operatorname{Span}(A)$ is finite.
   – Let $\mathcal{F}_B, \mathcal{F}_C$ be the sets of finite subsets of B, C respectively, and let $f \colon \mathcal{F}_B \to \mathcal{F}_C$ be the function $f(A) = C \cap \operatorname{Span}(A)$.
   (c) Show that the image of f covers C (in symbols, $\bigcup f(\mathcal{F}_B) = C$).
   (d) Show that the cardinality of the image of f is at least that of C.
   (e) Show that $|B| \geq |C|$. Conclude that $|B| = |C|$, in other words that infinite-dimensional vector spaces also have well-defined dimensions.