Math 100 – SOLUTIONS TO WORKSHEET 21
ANTIDERIVATIVES

1. Warmup: inverse operations

(1) (Multiplication)
  (a) Calculate: $7 \times 8 =$
  (b) Find (some) a, b such that $ab = 15$.
(2) (Trig functions)
  (a) Calculate: $\sin \frac{\pi}{3} =$
  (b) Find all θ such that $\sin\theta = 1$.
  Solution: $\frac{\pi}{2} + 2\pi\mathbb{Z}$ or $\left\{\frac{\pi}{2} + 2\pi k\right\}_{k\in\mathbb{Z}}$.
(3) Simple differentiation
  (a) Find one f such that $f'(x) = 1$.
  Solution: $f(x) = x$ works.
  (b) Find *all* such f.
  Solution: $\boxed{f(x) = x + C}$ where C is an arbitrary constant.
  (c) Find the f such that $f(7) = 3$.
  Solution: We need $7 + C = 3$ so $C = -4$ and hence $\boxed{f(x) = x - 4}$.

2. Antidifferentiation by massaging

(4) Find f such that $f'(x) = 2x^3$.

Solution: We know the derivative of x^4 is $4x^3$, so the derivative of $\boxed{\frac{1}{2}x^4}$ is $2x^3$ as desired.

(5) Find f such that $f'(x) = -\frac{1}{x}$.

Solution: We know the derivative of $\log|x|$ is $\frac{1}{x}$, so the derivative of $\boxed{-\log|x|}$ is $-\frac{1}{x}$.

(6) Find all f such that $f'(x) = \cos 3x$.

Solution: The derivative of $\sin x$ is $\cos x$, so the derivative of $\sin 3x$ is $3\cos 3x$ and the derivative of $\boxed{\frac{1}{3}\sin 3x}$ is $\cos 3x$.

3. Combinations

(7) (Final, 2015) Find a function $f(x)$ such that $f'(x) = \sin x + \frac{2}{\sqrt{x}}$ and $f(\pi) = 0$.

Solution: We know $(\cos x)' = -\sin x$. Also, $\left(x^{1/2}\right)' = \frac{1}{2\sqrt{x}}$.The general antiderivative is therefore

$$f(x) = -\cos x + 4\sqrt{x} + C\,.$$

To determine the constant we evaluate at π:

$$0 = f(\pi) = -\cos\pi + 4\sqrt{\pi} + C = 1 + 4\sqrt{\pi} + C\,.$$

We therefore have $C = -1 - 4\sqrt{\pi}$ and

$$\boxed{f(x) = -\cos x + 4\sqrt{x} + 1 - 4\sqrt{\pi}\,.}$$

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(8) (Final, 2016) Find the general antiderivative of $f(x) = e^{2x+3}$.

Solution: Write $f(x) = e^3 e^{2x}$. Since the derivative of e^x is e^x the derivative of e^{2x} is $2e^{2x}$ and $\boxed{f(x) = \frac{1}{2}e^3 e^{2x} + C}$.

(9) Find f such that $f'(x) = \frac{6x^4-2x-2}{x^2}$.

Solution: We have $\frac{6x^4-2x-2}{x^2} = 6x^2 - \frac{2}{x} - \frac{2}{x^2}$. Since the derivative of x^3 is $3x^2$, since the derivative of $\log|x|$ is $\frac{1}{x}$ and since the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$ we may use

$$\boxed{f(x) = 2x^3 - 2\log|x| + \frac{2}{x}\,.}$$

(10) Find f such that $f'(x) = 2x^{1/3} - x^{-2/3}$ and $f(1000) = 5$.

Solution: Since $\left(x^{4/3}\right)' = \frac{4}{3}x^{1/3}$ and $\left(x^{1/3}\right)' = \frac{1}{3}x^{-2/3}$ the general solutions is

$$f(x) = 2\cdot\frac{3}{4}x^{4/3} - 3x^{1/3} + c\,.$$

To get the specific solution we solve using $(1000)^{1/3} = 10$:

$$\begin{aligned} 5 &= f(1000) = \frac{3}{2}(1000)^{4/3} - 3(1000)^{1/3} + c \\ &= \frac{3}{2}10^4 - 30 + c \end{aligned}$$

so

$$c = 35 - 15,000 = -14,965$$

and

$$\boxed{f(x) = \frac{3}{2}x^{4/3} - 3x^{1/3} - 14,965\,.}$$

(11) Find f such that $f''(x) = \sin x + \cos x$, $f(0) = 0$ and $f'(0) = 1$.

Solution: Since $(f')'(x) = \sin x + \cos x$, $f'(x) = -\cos x + \sin x + c$. Now $f'(0) = -1 + 0 + c = 1$ so $c = 2$ and $f'(x) = -\cos x + \sin x + 2$. From this we get $f(x) = -\sin x - \cos x + 2x + d$ for some d. We also need $f(0) = -0 - 1 + 0 + d = 0$ so $d = 1$ and

$$\boxed{f(x) = -\sin x - \cos x + 2x + 1\,.}$$