Math 100 – SOLUTIONS TO WORKSHEET 20
L'HÔPITAL'S RULE

(1) Evaluate $\lim_{x\to1}\frac{\log x}{x-1}$.

Solution: Since $\lim_{x\to1}\log x=\log 1=0$ and $\lim_{x\to1} x-1=1-1=0$ and since both the numerator and denominator are differentiable we apply l'Hôpital's rule and get:

$$\lim_{x\to1}\frac{\log x}{x-1}=\lim_{x\to1}\frac{1/x}{1}=\lim_{x\to1}\frac{1}{x}=1\,.$$

(2) (Final, 2014) Evaluate $\lim_{x\to0}\frac{\cos x-e^{x^2}}{x^2}$.

Solution: We apply l'Hôpital's rule twice to get:

$$\begin{aligned}\lim_{x\to0}\frac{\cos x-e^{x^2}}{x^2}&=\lim_{x\to0}\frac{-\sin x-2xe^{x^2}}{2x}\\&=\lim_{x\to0}\frac{-\cos x-2e^{x^2}+4x^2e^{x^2}}{2}\\&=\frac{-\cos 0-2e^0+0}{2}=-\frac{3}{2}\,.\end{aligned}$$

The use of the rule is justified since $\lim_{x\to0}\left(\cos x-e^{x^2}\right)=\cos 0-e^0=0$, $\lim_{x\to0}\left(-\sin x-2xe^{x^2}\right)=-\sin 0-0=0$ and $\lim_{x\to0}x^2=\lim_{x\to0}2x=0$.

(3) Do (2) using a 2nd-order Taylor expansion.

Solution: To second order we have $\cos x\approx1-\frac{x^2}{2}$ and $e^u\approx1+u+\frac{u^2}{2}$ so $e^{x^2}\approx1+x^2$. We therefore have $\cos x-e^{x^2}\approx\left(1-\frac{x^2}{2}\right)-\left(1+x^2\right)=-\frac{3}{2}x^2$. We therefore have $\frac{\cos x-e^{x^2}}{x^2}\approx-\frac{3}{2}$ to zeroth order.

(4) (Final, 2015) Evaluate $\lim_{x\to0}\frac{\log(1+x)-\sin x}{x^2}$.

Solution: We apply l'Hôpital's rule twice to get:

$$\lim_{x\to0}\frac{\log(1+x)-\sin x}{x^2}=\lim_{x\to0}\frac{\frac{1}{1+x}-\cos x}{2x}=\lim_{x\to0}\frac{-\frac{1}{(1+x)^2}+\sin x}{2}=-\frac{1}{2}\,.$$

The use of the rule is justified since $\lim_{x\to0}\left(\log(1+x)-\sin x\right)=\log 1-\sin 0=0$, $\lim_{x\to0}\left(\frac{1}{1+x}-\cos x\right)=\frac{1}{1}-\cos 0=0$ and $\lim_{x\to0}x^2=\lim_{x\to0}2x=0$.

Remark: To third order we have $\log(1+x)\approx x-\frac{x^2}{2}+\frac{x^3}{3}$ and $\sin x=x-\frac{x^3}{6}$ so, to first order,

$$\frac{\log(1+x)-\sin x}{x^2}\approx\frac{\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)-\left(x-\frac{x^3}{6}\right)}{x^2}=\frac{-\frac{x^2}{2}+\frac{1}{2}x^3}{x^2}=-\frac{1}{2}+\frac{1}{2}x\xrightarrow[x\to0]{}\frac{1}{2}\,.$$

(5) Given that $f(2)=5$, $g(2)=3$, $f'(2)=7$ and $g'(2)=4$ find $\lim_{x\to3}\frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3}$.

Solution: Since f,g are differentiable at 2, they are continuous there and

$$\lim_{x\to3}\left(f(2x-4)-g(x-1)-2\right)=f(6-4)-g(3-1)-2=f(2)-g(2)-2=5-3-2=0$$
$$\lim_{x\to3}\left(g(x^2-7)-3\right)=g(9-7)-3=g(2)-3=3-3=0\,.$$

Date: 23/11/2021, Worksheet by Lior Silberman.

By arithmetic of derivatives the numerator and denominator are differentiable at $x = 3$ and we may therefore apply l'Hopital's rule:

$$\lim_{x\to 3} \frac{f(2x-4) - g(x-1) - 2}{g(x^2-7) - 3} = \lim_{x\to 3} \frac{2f'(2x-4) - g'(x-1)}{2xg'(x^2-7)}$$
$$= \frac{2f'(2) - g'(2)}{2\cdot 3\cdot g'(2)} = \frac{2\cdot 7 - 4}{6\cdot 4} = \frac{10}{24} = \frac{5}{12}\,.$$

(6) Evaluate $\lim_{x\to 0^+} \frac{e^x}{x}$.

Solution: Since $e^x \xrightarrow[x\to 0]{} e^0 = 1$ while $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ we have $\lim_{x\to 0^+} \frac{e^x}{x} = \infty$.

(7) Evaluate $\lim_{x\to\infty} x^2 e^{-x}$.

Solution: We have $x^2 e^{-x} = \frac{x^2}{e^x}$ and as $x \to \infty$ both numerator and denominator diverge to ∞. The same holds for $2x$. We may therefore apply l'Hôpital's rule twice and get:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

(8) Evaluate $\lim_{x\to 0^+} x\log x$.

Solution: We have $x\log x = \frac{\log x}{1/x}$ and as $x \to 0^+$ both numerator and denominator diverge ($\log x$ to $-\infty$, $\frac{1}{x}$ to ∞). We may therefore apply l'Hôpital's rule twice and get:

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0\,.$$

(9) Evaluate $\lim_{x\to 0} (2x+1)^{1/\sin x}$.

Solution: We have $(2x+1)^{1/\sin x} = e^{\frac{\log(2x+1)}{\sin x}}$. Now since the function e^u is continuous it's enough to compute $\lim_{x\to 0} \frac{\log(2x+1)}{\sin x}$. As $x \to 0$, $\log(2x+1) \to \log 1 = 0$ and $\sin x \to \sin 0 = 0$ so we may apply l'Hôpital's rule and get

$$\lim_{x\to 0} \frac{\log(2x+1)}{\sin x} = \lim_{x\to 0} \frac{\frac{2}{2x+1}}{\cos x} = \lim_{x\to 0} \frac{2}{(2x+1)\cos x} = \frac{2}{\cos 0} = 2\,.$$

We therefore have

$$\lim_{x\to 0} (2x+1)^{1/\sin x} = \lim_{x\to 0} e^{\frac{\log(2x+1)}{\sin x}} e^{\lim_{x\to 0} \frac{\log(2x+1)}{\sin x}} = e^2\,.$$

(10) Evaluate $\lim_{x\to\infty} x^n e^{-x}$.

Solution: We note that for every $k > 0$, $\lim_{x\to\infty} x^k = \infty$ and similarly $\lim_{x\to\infty} e^x = \infty$. We therefore apply l'Hôpital's rule n times to get:

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0\,.$$

(11) Suppose $a > 0$. Evaluate $\lim_{x\to\infty} x^{-a}\log x$.

Solution: We have $x^{-a}\log x = \frac{\log x}{x^a}$ and as $x \to \infty$ both numerator and denominator diverge to ∞. We may therefore apply l'Hôpital's rule get:

$$\lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{x\to\infty} \frac{1/x}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{ax^a} = 0$$

since $a > 0$.