Math 100 – SOLUTIONS TO WORKSHEET 14
TAYLOR EXPANSION

1. Taylor approximation

(1) (Review) Use linear approximations to estimate:
  (a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.
  Solution: Let $f(x) = \log x$ so that $f'(x) = \frac{1}{x}$. Then $f(1) = 0$ and $f'(1) = 1$ so $f(1+\frac{1}{3}) \approx \frac{1}{3}$ and $f(1-\frac{1}{3}) \approx -\frac{1}{3}$. Then $\log 2 = \log\frac{4}{3}/\frac{2}{3} = \log\frac{4}{3} - \log\frac{2}{3} \approx \frac{2}{3}$.
  (b) $\sin 0.1$ and $\cos 0.1$.
  Solution: Let $f(x) = \sin x$ so that $g(x) = f'(x) = \cos x$ and $g'(x) = -\sin x$. Then $f(1) = 0$ and $g(0) = f'(0) = \cos 0 = 1$ while $g'(0) = -\sin 0 = 0$. So $f(0.1) \approx 0 + 1\cdot 0.1 \approx 0.1$ and $g(0.1) \approx 1 - 0\cdot 0.01 = 1$.

(2) Let $f(x) = e^x$
  (a) Find $f(0), f'(0), f^{(2)}(0), \cdots$
  (b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.
  (c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.
  (d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.
  (e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.
  Solution: $f(x) = f'(x) = f^{(2)}(x) = \cdots = e^x$ so $f(0) = f'(0) = f''(0) = \cdots = 1$. Now $T_0(x) = 1$ works, as does $T_1(x) = 1 + x$. If $T_2(x) = 1 + x + cx^2$ then $T_2''(x) = 2c = 1$ means $c = \frac{1}{2}$ and $T_2(x) = 1 + x + \frac{1}{2}x^2$. Finally, $T_3(x) = 1 + x + \frac{1}{2}x^2 + dx^3$ works if $6d = 1$ so if $d = \frac{1}{6}$.

(3) Do the same with $f(x) = \ln x$ about $x = 1$.
  Solution: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ so $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$. Try $T_3(x) = a + bx + cx^2 + dx^3$ (can truncate later). Need $a = 0$ to make $T_3(x) = 0$. Diff we get $T_3'(x) = b + 2cx + 3dx^2$, setting $x = 0$ gives $b = 1$. Diff again gives $T_3''(x) = 2c + 6dx$ so $2c = -1$ and $c = -\frac{1}{2}$. Diff again give $T_3'''(x) = 6d = 2$ so $d = \frac{1}{3}$ and $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$. Truncate this to get T_0, T_1, T_2.

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)
  Solution: $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f^{(3)}(x) = \frac{6}{(1-x)^4}$, $f^{(4)}(x) = \frac{24}{(1-x)^5}$ $f^{(k)}(0) = k!$ and the Taylor expansion is $1 + x + x^2 + x^3 + x^4$.

(5) Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion
  Solution: $(\cos x)' = -\sin x$, $(\cos x)^{(2)} = -\cos x$, $(\cos x)^{(3)} = \sin x$, $(\cos x)^{(4)}(x) = \cos x$ and the pattern repeats. Plugging in zero we see that the derivatives at 0 (starting with the zeroeth) are $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ so the Taylor expansion is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

In particular, $\cos 0.1 \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$.

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?
  Solution: We have $c_2 = \frac{f^{(2)}}{2!} = 12$ so $f^{(2)} = 24$.

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(7) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x=0$.

Solution: Let $f(x)=\sin x$. Then $f'(x)=\cos x$, $f^{(2)}(x)=-\sin x$ and $f^{(3)}(x)=-\cos x$. Thus $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f^{(3)}(0)=-1$ and the third-order expansion of $\sin x$ is $0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{(-1)}{3!}x^3=x-\frac{1}{6}x^3$. In particular $\sin 0.1\approx 0.1-\frac{1}{6000}$. We then also have, correct to third order, that

$$(x+1)\sin x\approx(x+1)\left(x-\frac{1}{6}x^3\right)=x+x^2-\frac{1}{6}x^3-\frac{1}{6}x^4\approx x+x^2-\frac{1}{6}x^3\,.$$

(8) Find the 3rd order Taylor expansion of $\sqrt{x}+3x$ about $x=4$.

Solution: Let $f(x)=\sqrt{x}$. Then $f'(x)=\frac{1}{2\sqrt{x}}$, $f^{(2)}(x)=-\frac{1}{4x^{3/2}}$ and $f^{(3)}(x)=\frac{3}{8}x^{-5/2}$. Thus $f(4)=2$, $f'(4)=\frac{1}{4}$, $f^{(2)}(4)=-\frac{1}{32}$, $f^{(3)}(4)=\frac{3}{256}$ and the third-order expansions are

$$\begin{aligned}\sqrt{x}&\approx 2+\frac{1}{4}(x-4)-\frac{1}{32\cdot 2!}(x-4)^3+\frac{3}{256\cdot 3!}(x-4)^3\\ 3x&\approx 12+3(x-4)\end{aligned}$$

so that

$$\sqrt{x}+3x\approx 14+3\frac{1}{4}\cdot(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3\,.$$

(9) Find the 8th order expansion of $f(x)=e^{x^2}+\cos(2x)$. What is $f^{(6)}(0)$?

Solution: To fourth order we have $e^u=1+u+\frac{u^2}{2}+\frac{u^3}{6}+\frac{u^4}{24}$ so $e^{x^2}=1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\frac{x^8}{24}$. We also know that $\cos u=1-\frac{u^2}{2}+\frac{u^4}{24}-\frac{u^6}{720}+\frac{u^8}{40320}$ so $\cos(2x)=1-2x^2+\frac{2}{3}x^4-\frac{4}{45}x^6+\frac{2}{315}x^8$ so

$$\begin{aligned}e^{x^2}+\cos(2x)&\approx\left(1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\frac{x^8}{24}\right)+\left(1-2x^2+\frac{2}{3}x^4-\frac{4}{45}x^6+\frac{2}{315}x^8\right)\\ &=2-x^2+\frac{7}{6}x^4+\frac{7}{90}x^6+\frac{121}{2520}x^8\,.\end{aligned}$$

In particular, $\frac{f^{(6)}(0)}{6!}=\frac{7}{90}$ so $f^{(6)}(0)=6!\cdot\frac{7}{90}=\frac{720\cdot 7}{90}=56$.

(10) Show that $\log\frac{1+x}{1-x}\approx 2(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x.

Solution: Let $f(x)=\log(1+x)$. Then $f'(x)=\frac{1}{1+x}$, $f^{(2)}(x)=-\frac{1}{(1+x)^2}$, $f^{(3)}(x)=\frac{1\cdot 2}{(1+x)^3}$, $f^{(4)}(x)=-\frac{1\cdot 2\cdot 3}{(1+x)^4}$ and so on, so $f^{(k)}(x)=(-1)^{k-1}\cdot\frac{(k-1)!}{(1+x)^k}$. We thus have that $f(0)=0$ and for $k\ge 1$ that $f^{(k)}(0)=(-1)^{k-1}(k-1)!$. Then $\frac{f^{(k)}(0)}{k!}=\frac{(-1)^{k-1}}{k}$ so

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\,.$$

Plugging $-x$ we get:

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}\cdots$$

so

$$\log\frac{1+x}{1-x}=\log(1+x)-\log(1-x)=2x+2\frac{x^3}{3}+2\frac{x^5}{5}+\cdots\,.$$

In particular

$$\log 3=\log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=2\left(\frac{1}{2}+\frac{1}{24}+\frac{1}{160}+\cdots\right)=1+\frac{1}{12}+\frac{1}{80}+\cdots\approx 1.096$$