15. Taylor Remainder (2/11/2021)

Goals.

(1) Review Taylor expansion
(2) Lagrange remainder for linear approximation
(3) Lagrange remainder: general case

Last Time. Taylor expansion

Give function f, point a, can make: (for x "near" a)

0) constant approx: $f(x) \approx f(a)$ ← "continuity"

1) linear approx: $f(x) \approx f(a) + f'(a)(x-a)$ ← "derivative" = "tangent line"

2) quadratic approx: $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

3) cubic approx: $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{6}(x-a)^3$

⋮

n'th order correction is $\frac{f^{(n)}(a)}{n!}(x-a)^n$

$n!$ = "n factorial" $= 1\cdot2\cdot3\cdot\ldots\cdot n$; $0! = 1! = 1$

Example: Formula from special relativity

Energy → $E(v) = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

m = mass of body
c = speed of light
V = velocity

(only makes sense if $|v| < c$,
also $\lim_{v\to c} E(v) = \infty$)

Goal: practise Taylor expansion

Physics: understand $\epsilon(v)$ for small velocities

let $x = \frac{v^2}{c^2}$ then v small $\leftrightarrow$ x small. Look at

$$\epsilon(v) = \frac{mc^2}{\sqrt{1-x}} = mc^2 (1-x)^{-\frac{1}{2}}$$

$\epsilon(0) = mc^2$, $\quad \epsilon'(x) = mc^2 \cdot (-\frac{1}{2}) \cdot (1-x)^{-3/2} (-1)$

$\epsilon'(0) = +\frac{1}{2} mc^2$

$$\epsilon''(x) = \frac{1}{2} mc^2 \cdot (-\frac{3}{2}) (1-x)^{-5/2} (-1) = \frac{3}{4} mc^2 (1-x)^{-5/2}$$

$\epsilon''(0) = \frac{3}{4} mc^2 \quad \ldots$

Taylor expansion

so $\epsilon(x) \approx \underbrace{mc^2}_{\epsilon(0)} + \underbrace{\frac{1}{2} mc^2}_{\epsilon'(0)} x + \underbrace{\frac{1}{2}}_{1/2!} \underbrace{\frac{3}{4} mc^2}_{\epsilon^{(2)}(0)} x^2 + \ldots$

so $\epsilon(v) \approx mc^2 + \frac{1}{2} mc^2 \frac{v^2}{c^2} + \frac{3}{8} mc^2 \left(\frac{v^2}{c^2}\right)^2 + \ldots$

$= mc^2 + \frac{1}{2} mv^2 + \frac{3}{8} mc^2 \left(\frac{v^2}{c^2}\right)^2 + \ldots$

Today: How far is the approximation from the truth?

Example: (MVT): How far is constant from truth?

$$\underbrace{f(x)}_{\text{truth}} = f(a) + \underbrace{f'(c)}_{\text{Const}}(x-a)$$

error/difference/remainder

c between a, x.

Math 100 – WORKSHEET 15
TAYLOR REMAINDER ESTIMATES

1. Review: Taylor expansion

(1) Estimate $(4.1)^{3/2}$ using a linear and a quadratic approximation.

let $f(x) = x^{3/2}$, $a=4$; $f'(x) = \frac{3}{2}x^{1/2}$, $f''(x) = \frac{3}{4}x^{-1/2}$

$f(4) = 8$, $f'(4) = 3$, $f''(4) = \frac{3}{8}$ so

to 1^{st} order: $f(4.1) \approx 8 + 3(0.1) = 8.3$

to 2^{nd} order $f(4.1) \approx 8 + 3(0.1) + \frac{1}{2}\cdot\frac{3}{8}\cdot(0.1)^2 = 8.3 + \frac{3}{1600}$

(2) The third-order expansion of $h(x)$ about $x = 2$ is $3 + \frac{1}{2}(x-2) + 2(x-2)^3$. What are $h'(2)$ and $h''(2)$?

$h'(2)$ = slope of linear term $= \frac{1}{2}$

$h''(2) = 0$ (no quadratic term)

Date: 2/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$T_1(x) = f(a) + f'(a)(x-a)$ c between a, x

$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(c)(x-a)^2$

2. Error estimate 1

Let $R_1(x) = f(x) - T_1(x)$ be the *remainder*. Then there is c between a and x such that

$$R_1(x) = \frac{f^{(2)}(c)}{2!}(x-a)^2$$

We found $(4.1)^{3/2} \approx \cancel{4}8.3$ to first order.

(4) Estimate the error in the linear approximations to $(4.1)^{3/2}$.

$$R_1(4.1) = \underbrace{\frac{1}{2}}_{\frac{1}{2!}} \cdot \underbrace{\frac{3}{4} c^{-1/2}}_{f^{(2)}(c)} \underbrace{(4.1-4)^2}_{(x-a)^2}$$

looks like quadratic term in Taylor expansion

true value: $(4.1)^{3/2}$

linear approx: 8.3

formula: $(4.1)^{3/2} - 8.3 = \frac{3}{800} c^{-1/2}$ for some $4 < c < 4.1$

can't determine c exactly. But can tell: $\frac{3}{800} c^{-1/2} > 0$

so $(4.1)^{3/2} > 8.3$ ("8.3 is an underestimate")

also $\frac{3}{800} c^{-1/2} < \frac{3}{800} \cdot \frac{1}{2} = \frac{3}{1600}$ if $c > 4$, $c^{-\frac{1}{2}} < 4^{-1/2} = \frac{1}{2}$

so $(4.1)^{3/2} - 8.3 < \frac{3}{1600}$

ie $8.3 < (4.1)^{3/2} < 8.3 + 3/1600$

Recap: (1) Error in linear approx looks like the quadratic term except with $f^{(2)}(c)$ rather that $f^{(2)}(a)$.

(2) Can use "c between a, x" to give bounds on $f^{(2)}(c) \Rightarrow$ info on $f(x)$

Example: $f(x) = \log(1-x^2)$ want info on $\log\left(\frac{8}{9}\right) = f\left(\frac{1}{3}\right)$

(1) linear approx: $f'(x) = \dfrac{-2x}{1-x^2}$

$f(0) = \log 1 = 0, \quad f'(0) = 0 \qquad f\left(\frac{1}{3}\right) \approx \underset{f(a)}{0} + \underset{f'(a)}{0}\underset{x-a}{\left(\frac{1}{3}\right)} = 0.$

(2) so $f\left(\frac{1}{3}\right) = 0 + R_1\left(\frac{1}{3}\right) = \frac{1}{2}\cdot\left(-2\frac{1+c^2}{(1-c^2)^2}\right)\cdot\left(\frac{1}{3}-0\right)^2 =$

$$f''(x) = \frac{-2(1-x^2)-(-2x)(-2x)}{(1-x^2)^2} = -\frac{2+2x^2}{(1-x^2)^2} = -2\frac{1+x^2}{(1-x^2)^2}$$

$$f\left(\tfrac{1}{3}\right) = -\frac{1}{9}\cdot\frac{1+c^2}{(1-c^2)^2}$$

note: $\dfrac{1+c^2}{(1-c^2)^2}$ is increasing with c, so $\left(0<c<\frac{1}{3}\right)$

$$1 = \frac{1+0^2}{(1-0)^2} < \frac{1+c^2}{(1-c^2)^2} < \frac{1+\left(\frac{1}{3}\right)^2}{(1-(1/3)^2)^2} = \frac{10/9}{(8/9)^2}$$

so

$$f\left(\tfrac{1}{3}\right) < -\frac{1}{9}$$

$$f\left(\tfrac{1}{3}\right) > -\frac{1}{9}\cdot\frac{10/9}{(8/9)^2} = -\frac{10}{8^2} = -5/32.$$

3. Higher order error estimates

Let $R_n(x) = f(x) - T_n(x)$ be the *remainder*. Then there is c between a and x such that
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

We found $(4.1)^{3/2} \approx \cancel{4}8.301875$ to second order.

(6) Estimate the magnitude of the error in the quadratic approximation to $(4.1)^{3/2}$.

$f(x) = x^{3/2} \quad f'(x) = \frac{3}{2}x^{1/2}, \quad f^{(2)}(x) = \frac{3}{4}x^{-1/2}, \quad a = 4$

quad approx: $T_2(4.1) = 8 + 0.3 + \frac{3}{1600} = 8.301875$

<u>Error:</u> $f^{(3)}(x) = -\frac{3}{8}x^{-3/2}$

so $R_2(4.1) = \frac{1}{3!}\left(-\frac{3}{8}c^{-3/2}\right)\cdot(4.1-4)^3, \quad 4 < c < 4.1$

$$= -\frac{3}{48{,}000}c^{-3/2} = -\frac{1}{16{,}000}c^{-3/2}$$

so $|R_2(4.1)| < \frac{1}{16{,}000}\cdot 4^{-3/2} = \frac{1}{128{,}000}$

<u>aside:</u> $R_2(4.1) < 0$ so $T_2(4.1) > f(4.1)$: 8.301875 is an <u>over</u> estimate

$-\frac{1}{128{,}000} < R_2(4.1) < 0$ so $|R_1(4.1)| < \frac{1}{128{,}000}$

Did not choose $c = 4$

True: $c > 4$ so $c^{-3/2} < \frac{1}{8}$

$c < 4.1$ so $c^{-3/2} > \frac{1}{(4.1)^{3/2}}$ ← true, not useful

$$|R_2(4,1)| = \frac{1}{16{,}000} c^{-3/2} < \frac{1}{128{,}000}$$

Example: say $f^{(4)}(x) = \frac{\cos(x^2)}{3-x}$

Estimate $R_3(0.5) = \frac{1}{24} \frac{\cos(c^2)}{3-c} \cdot \left(\frac{1}{2}\right)^4$

note: $\left\{ \begin{array}{l} \cos(c^2) \le 1 \\ \frac{1}{3-c} \le \frac{1}{3-\frac{1}{2}} \end{array} \right.$ $0 < c < \frac{1}{2}$

$\Downarrow$

$$\frac{\cos(c^2)}{3-c} \le \frac{1}{2.5}$$