13. The Mean Value Theorem (26/10/2021)

Goals.

(1) The Mean Value Theorem
(2) The Linear approximation

Last Time.

Exponential growth & decay: $N(t) = Ce^{kt}$

($k>0$ for growth, $k<0$ for decay)

Don't have to use base e. For example: $N_0 \cdot \left(\frac{1}{2}\right)^{t/\tau}$

(τ = half-life) but:

$$2^{-t/\tau} = \left(\frac{1}{2}\right)^{t/\tau} = e^{-\frac{\log 2}{\tau} t}.$$

Example: NLC: exponential decay of temperature difference to the environment.

(often arises from models of the form $y' = ky$)

Suppose I drive 60 km during 1 hr in my car.

Average speed: $60 \frac{km}{hr}$

Could I have always been driving at velocity $> 60 \frac{km}{h}$?

" " " " " " " " $< 60 \frac{km}{h}$?

$\Rightarrow$ At some point I must have been going at $60 \frac{km}{h}$ exactly.

Math 100 – WORKSHEET 17
THE MEAN VALUE THEOREM; LINEAR APPROXIMATION

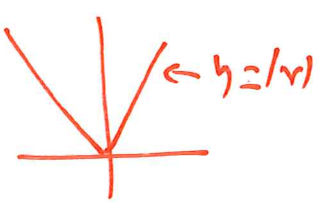
1. Average slope vs Instantenous slope

(1) Let $f(x) = e^x$ on the interval $[0,1]$. Find all values of c so that $f'(c) = \frac{f(1)-f(0)}{1-0}$.

need to solve $e^c = \frac{e-1}{1-0} = e-1$ so $c = \log(e-1)$

(2) Let $f(x) = |x|$ on the interval $[-1,2]$. Find all values of c so that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$.

$$f'(c) = \begin{cases} -1 & c<0 \\ \text{undef} & c=0 \\ +1 & c>0 \end{cases} \qquad \frac{f(2)-f(-1)}{2-(-1)} = \frac{2-1}{3} = \frac{1}{3}$$



Date: 26/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Mean value theorem

<u>Fact:</u> If f is <u>diff</u> on $[a,b]$ then have $c \in (a,b)$

s.t. $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

("instantaneous slope at c = average slope on $[a,b]$")

<u>Takeaways:</u> (1) must check differentiability.

(2) if can solve for c don't need thm

Worksheet (3)

<u>Example:</u> Suppose $f'(x) > 0$ on (a,b)

≥ 0 on (a,b)

then $f(b) > f(a)$

$f(b) \geq f(a)$.

<u>Proof:</u> $$\frac{f(b)-f(a)}{b-a} = f'(c) \begin{matrix} > 0 \\ \geq 0 \end{matrix}$$ for some $c \in (a,b)$ by MVT

mult by $b-a$, get $f(b) - f(a) > 0$

or $f(b) - f(a) \geq 0$.

2. The Mean Value Theorem

(3) Show that $f(x) = 3x^3 + 2x - 1 + \sin x$ has exactly one real zero. (Hint: let $a < b$ be zeroes of f. The MVT will find c such that $f'(c) = ?$)

Suppose we had two different zeroes $a < b$.

f is diff everywhere (defined by formula everywhere)

So by MVT there is $c \in (a,b)$ s.t.:

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$$

But $f'(x) = 9x^2 + 2 + \cos x \geq 0 + 2 - 1 = 1 > 0$

So there is no such c, so no such a, b

(4) (Final, 2015)

(a) Suppose f, f', f'' are all continuous. Suppose f has at least three zeroes. How many zeroes must f', f'' have?

If $a<b$ are ~~zeroes~~ zeroes of f, by MVT have $a<c<b$ s.t. $f'(c)=0$ (f is diff here) so if $a<b<c$ are zeroes of f, get zero of f' (call it d) between a,b get zero of f' (call it e) between b,c

For same reason, between d,e have zero of $(f')'=f''$ (call it g)

(b) [Show that $2x^2-3+\sin x+\cos x=0$ has at least two solutions] ← use IVT

(c) Show that the equation has at most two solutions.

~~If not~~ let $f(x)=2x^2-3+\sin x+\cos x$.

If f ~~does~~ did not have at most two roots, it would have at least three. By part (a), f'' will have a root.

But $f''(x)=4-\sin x-\cos x \geq 4-1-1=2>0$

so $f''(x)$ has no roots, so f doesn't have three.

(5) (Final, 2012) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x \in [1, 4]$. What can you say about $f(4)$?

f is diff so by MVT, $\frac{f(4)-f(1)}{4-1} = f'(c)$ for some $c \in (1,4)$. So

$$-3 \leq \frac{f(4)-3}{3} \leq 2$$

So

$$-6 = \underset{f(1)}{3} + \underset{4-1}{3}\,\underset{\text{slope}}{(-3)} \leq f(4) \leq \underset{f(1)}{3} + \underset{\text{slope}}{2} \cdot \underset{4-1}{3} = 9$$

(6) Show that $|\sin a - \sin b| \leq |a - b|$ for all a, b.

(7) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1 + x) < x$.

Linear Approximation

MVT: Suppose we start a, go to x. Then have c between a, x s.t. $\frac{f(x) - f(a)}{x - a} = f'(c)$

i.e. $\underset{\substack{\uparrow \\ \text{value at } x}}{f(x)} = \underset{\substack{\uparrow \\ \text{value at } a}}{f(a)} + \underbrace{f'(c)(x-a)}_{\text{correction}}$ ← exact, but don't know c

if x close to a expect $f(x)$ close to $f(a)$
($\lim_{x \to a} f(x) = f(a)$ is continuity)

linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$
(hope: $f'(a)$ not too far from $f'(c)$)

(write tangent line $y = f(a) + f'(a)(x-a)$
evaluate at point x)

3. The Linear Approximation

(8) Use a linear approximation to estimate

(a) $\sqrt{1.2}$

Try $f(x) = \sqrt{x}$, $a = 1$. $f(1) = \sqrt{1} = 1$, $f'(1) = \left[\frac{1}{2\sqrt{x}}\right]_{x=1} = \frac{1}{2}$

~~approx~~ so $\underbrace{\sqrt{1.2}}_{f(x)} \approx \underbrace{\sqrt{1}}_{f(a)} + \underbrace{\frac{1}{2}}_{f'(a)}(\underbrace{1.2 - 1}_{x-a}) \approx 1.1$

(b) (Final, 2015) $\sqrt{8}$

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $a = 9$

$f(9) = \sqrt{9} = 3$, $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$,

So $f(8) \approx f(9) + f'(9)(8 - 9) = 3 + \frac{1}{6} \cdot (-1) = 2\frac{5}{6}$