12. Exponetial growth and decay (21/10/2021)

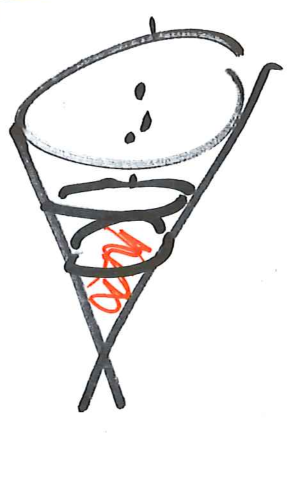
Goals.

(1) More related rates
(2) Exponential growth
(3) Exponential decay: half-life
(4) Newton's law of cooling

Last Time. inverse trig

related rates: can diff relation $F(x,y)=0$ wrt t to relate $\frac{dx}{dt}, \frac{dy}{dt}, x, y$.

Examples



(b) The drain is unclogged and water begins to clear at the rate of $\frac{\pi}{4}\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

Can also solve $\frac{dV}{dt} = \frac{\pi}{36} h^2 \frac{dh}{dt}$

for h or for $\frac{dh}{dt}$.

Q: at the time that $h = 5m$

$r = \frac{5}{6} m$

Why not: $V = \frac{1}{3}\pi\left(\frac{5}{6}\right)^2 h$

$$\frac{dV}{dt} = \frac{\pi}{3} \cdot 2rh \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

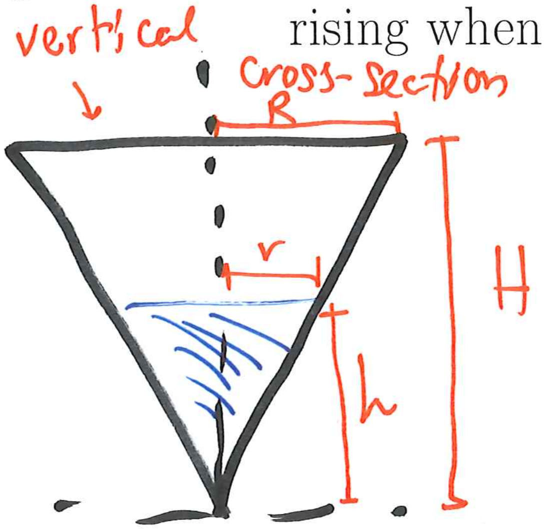
$$\frac{dr}{dt} = \frac{1}{6}\frac{dh}{dt}$$

1. More Related Rates

(1) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(0) read draw

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?



(1) names: R = radius of top of tank
H = height of tank
h = height of water
V = volume of water
r = radius of top of water

(2) relations: $V = \frac{1}{3}(\pi r^2)h$ (volume of cone)

$\frac{r}{h} = \frac{R}{H} = \frac{1}{6}$ (similar triangles)

CLP appendix for school facts

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{6}\right)^2 h = \frac{\pi}{108}h^3$

(3) calculus: $\frac{dV}{dt} = ?$

$\frac{dV}{dt} = \frac{\pi}{108}\cdot 3h^2\frac{dh}{dt}$

(4) endgame: $\frac{dh}{dt} = \left(\frac{36}{\pi\cdot 5^2}\cdot 5\right)\frac{m}{min}$

$= \frac{36}{5\pi}\ \frac{m}{min}$

Date: 21/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Related Rates Summary

(0) Read problem: understand the idea, draw a picture if possible.

(1) Assign names:
- Choose axes, quantities of interest.
- Give a *name* to each quantity of interest.

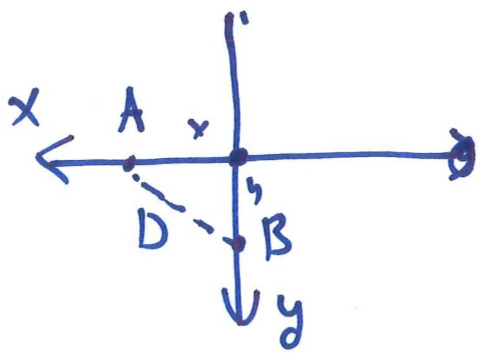
(2) Function: write down the *relation* between the quantities of interest.

(3) Calculus: differentiate the relation using the chain rule

(4) Interpretation: solve the problem using the calculus result.
- Make *sanity checks* (area can't be negative, for example).

(2) Two ships are travelling near an island. The first is located 20km due west of it, The second is located 15km due south of it and is moving due south at 7km/h. How fast is the distance between the ships changing if:

(a) The first ship is moving due ~~north~~ West at 5km/h.



ship A at x distance : D

ship B at y

$$D^2 = x^2 + y^2$$

$$\Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$x = 20\text{km}$ $\frac{dx}{dt} = 5\frac{\text{km}}{h}$

$y = 15\text{km}$ $\frac{dy}{dt} = 7\text{km}/h$

(b) The same setting, but now the first ship is moving toward the island.

now $\frac{dx}{dt} = -5\frac{\text{km}}{h}$

Exponential growth & Decay

1. motivation
2. examples
3. NLC.

Simple population model: At time t have $y(t)$ people in a short interval Δt, births & deaths happen at constant rate. net population change is: $\Delta y \approx y(t) \cdot r \cdot \Delta t$

(r = rate of growth / decay): $y(t+\Delta t) \approx y(t) + y(t) \cdot r \cdot \Delta t$

$$\Longleftrightarrow \quad \frac{y(t+\Delta t) - y(t)}{\Delta t} \approx r \cdot y(t)$$

take $\Delta t \to 0$ get $y'(t) = r\,y(t)$

Solution: $(e^t)' = e^t$ so $(e^{rt})' = r e^{rt}$

i.e. $y(t) = C e^{rt} = C (e^r)^t$

interpretation: $C = y(0)$

2. Exponential growth and decay

(3) Suppose[1] that a pair of invasive opossums arrives in BC in 1935. Unchecked, opossums can triple their population annually.

(a) At what time will there be 1000 opossums in BC? 10,000 opossums?

Let $N(t)$ be the number of opossums, t years after 1935.

$$N(t) = 2 \cdot 3^t = 2 \cdot e^{(\log 3) \cdot t}$$

So $N(t) = 1000$ when $2 \cdot e^{(\log 3) t} = 1000$, ie $t = \frac{\log 500}{\log 3} =$

$= \log_3 500$

(b) Write a differential equation expressing the growth of the opossum population with time.

[1]See http://linnet.geog.ubc.ca/efauna/Atlas/Atlas.aspx?sciname=Didelphis%20virginiana

Exponential decay often done in base 2:

Write $n(t) = n_0 \cdot 2^{-t/\tau}$

$= n_0 \cdot \left(\frac{1}{2}\right)^{t/\tau}$

τ = half-life = time it takes for n to halve

(4) A radioactive sample decays according to the law

$$\frac{dm}{dt} = km \,.$$

(a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?

(b) A 100-gram sample is left unattended for three days. How much of it remains?

(a)

If two half lives are 10hrs, one is 5h.

(b) after 3 days have $\left(\frac{1}{2}\right)^{72/5} \cdot 100\,gr$

$= e^{-\frac{\log 2}{5} \cdot 72} \cdot 100\,gr.$

(5) (Final, 2015) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many were present initially? Simplify your answer.

$n(t) = n(0) \cdot 2^{t/4}$ have $n(6)$ solve for $n(0)$

3. Newton's Law of Cooling

Fact. *When a body of temperature T_0 is placed in an environment of temperature T_{env} the temperature difference $T(t) - T_{env}$ between the body and the environment decays exponentially. In other words, there is a (negative) constant k such that*

$$T' = k(T - T_{env}) \qquad\qquad T(t) - T_{env} = (T_0 - T_{env})e^{kt}.$$

- *key idea*: change variables to the *temperature difference*. Let $y = T - T_{\text{env}}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} - 0 = ky$$

Corollary. $\lim_{t\to\infty} y(t) = 0$ *so* $\lim_{t\to\infty} T(y) = T_{env}$.