9. The Chain Rule; Inverse functions (12/10/2021)

Goals.

(1) Composition of functions
(2) The chain rule
(3) The inverse function rule

Last Time.

$$\frac{d}{dx} 2^x = (\log 2) 2^x \quad ; \quad \frac{d}{d\theta}\sin\theta = \cos\theta$$

$$\frac{d}{d\theta}\tan\theta = 1+\tan^2\theta = \frac{1}{\cos^2\theta} \quad ; \quad \frac{d}{d\theta}\cos\theta = -\sin\theta$$

(units: $\log x = \log_e x$; θ measured in radians)

$\sin 0 = \sin \pi = 0$, $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{2} = 1$

---

f is the composition of g & h if

$$f(x) = g(h(x))$$

suggestion: write g as a function of u, not x.

chain rule: if change x to $x+\Delta x$, $\Delta h \approx h'(x)\Delta x$

so $\Delta f \approx g'(h(x))\Delta h \approx g'(h(x)) \cdot h'(x) \cdot \Delta x$

Math 100 – WORKSHEET 9
THE CHAIN RULE; INVERSE FUNCTIONS

1. The chain rule

(1) Write the function as a composition and then differentiate.

(a) e^{3x}

① $e^{3x} = \cancel{f}(\cancel{h}(x))$ $g(u) = e^u$, $h(x) = 3x$

so $\frac{d(e^{3x})}{dx} = e^u \cdot 3 = 3e^{3x}$ $e^{3(x+\Delta x)} \approx e^{3x} + (3e^{3x})\Delta x$

② $e^{3x} = e^u$ where $u = 3x$ so

$\frac{d(e^{3x})}{dx} = \frac{d(e^{3x})}{du} \cdot \frac{du}{dx} = e^u \cdot 3 = 3e^{3x}$

(alternative solutions)

(b) $\sqrt{2x+1}$

$\sqrt{2x+1} = \sqrt{u}$ $u = 2x+1$

$\left(\sqrt{2x+1}\right)' = \frac{1}{2\sqrt{2x+1}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$

$\frac{d\sqrt{u}}{du}$ $\frac{d(2x+1)}{dx}$

Date: 12/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) (Final, 2015) $\sin(x^2)$

① $(\sin(x^2))' = \cos(x^2)\cdot 2x$

② let $\theta = x^2$ then $\sin(x^2) = \sin\theta$ so

$$\frac{d(\sin(x^2))}{dx} = \frac{d\sin\theta}{d\theta}\cdot\frac{d\theta}{dx} = \cos\theta\cdot(2x)$$

$$= \boxed{\cos(x^2)\cdot 2x}$$

go back to x at the end

(d) $(7x + \cos x)^n$.

$$\frac{d}{dx}(7x+\cos x)^n = n(7x+\cos x)^{n-1}\cdot(7-\sin x)$$

$[y = 7x+\cos x]$ $\quad \frac{d(y^n)}{dy} \quad \frac{dy}{dx}$

$$\frac{d}{d\theta}(\sin^2\theta) = \frac{d(\sin^2\theta)}{d(\sin\theta)}\cdot\frac{d(\sin\theta)}{d\theta} = (2\sin\theta)(\cos\theta) = \sin(2\theta)$$

Observe: $\frac{d}{d\theta}((\sin\theta)^2) \neq \frac{d}{d\theta}(\sin(\theta^2))$

(2) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

By the chain rule,

$$f'(x) = g'(2\sin x)\cdot(2\cos x)$$

so $f'\left(\frac{\pi}{4}\right) = g'\left(2\cdot\sin\frac{\pi}{4}\right)\cdot\left(2\cos\frac{\pi}{4}\right) = g'\left(2\cdot\frac{1}{\sqrt{2}}\right)\cdot\left(2\cdot\frac{1}{\sqrt{2}}\right)$

$= g'(\sqrt{2})\cdot\sqrt{2} = \sqrt{2}\cdot\sqrt{2} = 2.$

(3) Differentiate

(a) $7x + \cos(x^n)$

$$(7x+\cos(x^n))' = 7 - \sin(x^n)\cdot nx^{n-1}.$$

in detail: $(7x+\cos(x^n))' = 7(x)' + (\cos(x^n))'$ linearity

chain rule

$$= 7 + (-\sin(x^n))\cdot(nx^{n-1})$$

(b) $e^{\sqrt{\cos x}}$

$$\left(e^{\sqrt{\cos x}}\right)' = e^{\sqrt{\cos x}}\left(\sqrt{\cos x}\right)' = e^{\sqrt{\cos x}}\frac{1}{2\sqrt{\cos x}}(\cos x)'$$

$$= -e^{\sqrt{\cos x}}\cdot\frac{\sin x}{2\sqrt{\cos x}}$$

$$\frac{d(\sqrt{u})}{du} = \frac{d(u^{\frac{1}{2}})}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

(c) (Final 2012) $e^{(\sin x)^2}$

$$\frac{d}{dx}\left(e^{(\sin x)^2}\right) = e^{(\sin x)^2}\cdot 2\sin x\cdot\cos x = e^{\sin^2 x}\cdot\sin(2x)$$

Q: what about $x^{\sin^2 x}$? A: TBD

(4) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

If $f(g(x)) = x^3$ then $(f(g(x)))' = (x^3)'$

$f'(g(x)) \cdot g'(x) = 3x^2$

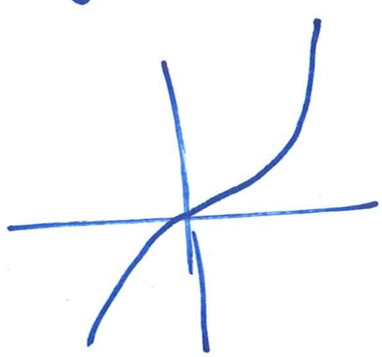
So $f'(g(4)) \cdot g'(4) = 3 \cdot 4^2$ so $g'(4) = \frac{48}{5}$

2. Inverse functions

(5) Find the function inverse to $y = x^7 + 3$.

If $y = x^7 + 3$, $x^7 = y - 3$ so $x = (y-3)^{1/7}$

(Aside: because 7 is odd, x^7 is increasing for all x so $y^{1/7}$ makes sense for all y)



i.e. we accept $y^{\frac{1}{7}}$ for negative y.

(6) Does $y = x^2$ have an inverse?

Inverse functions

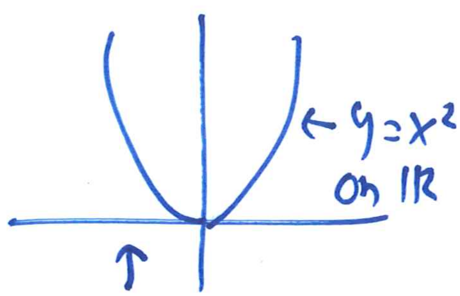
If $y = f(x)$ can find y for each x.

Sometimes we want reverse: x from y.

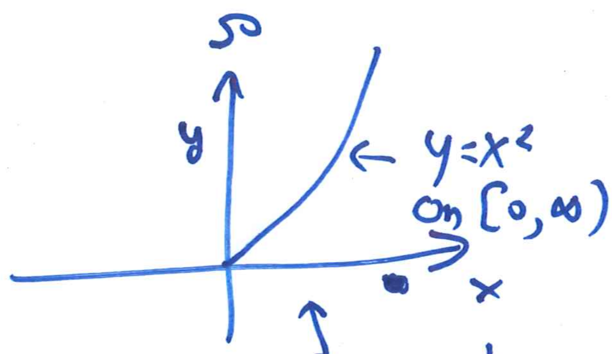
The inverse doesn't have to be a function: maybe multivalued. Eg: if $x^2 = y$ usually have two solutions for x.

Fact/Defn: Inverse function exists if for each y have one x value s.t. $y = f(x)$

Examples:



no inverse: two x-values for each y



$x = \sqrt{y}$ is the inverse

(7) Consider the function $y = \sqrt{x-1}$ on $x \geq 1$.

(a) Find the inverse function, in the form $x = g(y)$.

If $y = \sqrt{x-1}$, $x = 1+y^2$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}} \qquad \frac{dx}{dy} = 2y$$

so $\boxed{\frac{dy}{dx} \cdot \frac{dx}{dy} = 1}$

inverse function rule.

(need to keep x, y in the original meaning!)

(b) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$ and calculate their product.

(8) Let $f(x) = \log x$. Apply the chain rule to the formula $f(e^y) = y$ to get a formula for $f'(e^y)$, and use that to determine the derivative of the logarithm.

If $y = \log x$ then $x = e^y$

($\log x$ makes sense if x in range of e^y, i.e. if $x > 0$)

($\sqrt{y}$ makes sense if $y \geq 0$ because if $y = x^2$, $y \geq 0$)

going back to x

so $\frac{dx}{dy} = e^y$ so $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ i.e. $\frac{d(\log x)}{dx} = \frac{1}{x}$

inverse fcn rule

(9) Let $f(x) = x^3 + 5x$. Find $f^{-1}(6)$ and $\left(f^{-1}\right)'(6)$.