Math 100, lecture 23, 2/12/2021

Review 2

Q: How to evaluate $\lim_{x\to\infty} xe^{\frac{1}{x}} - x$?

note: $\lim_{x\to\infty} e^{\frac{1}{x}} = e^0 = 1$ so write $xe^{\frac{1}{x}} = x(e^{\frac{1}{x}} - 1)$

indeterminate form of the form $\infty \cdot 0$

$$= \frac{e^{\frac{1}{x}} - 1}{1/x} \underset{\uparrow}{=} \frac{e^y - 1}{y}$$

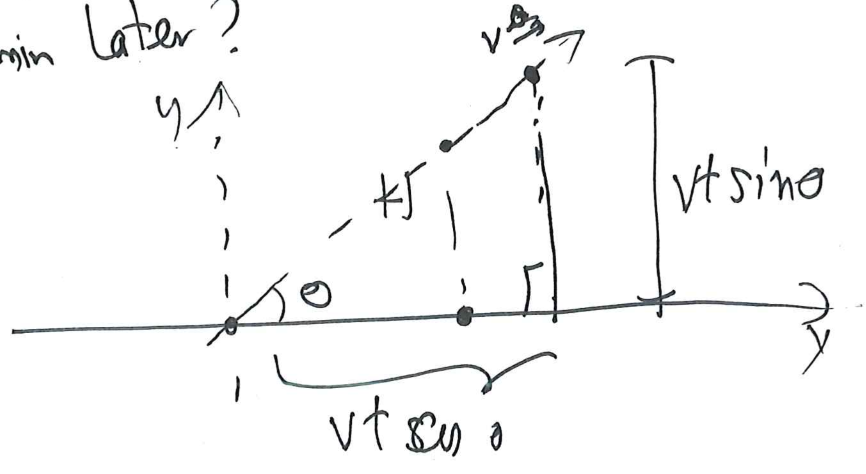
$y = \frac{1}{x}$

so $\lim_{x\to\infty} xe^{\frac{1}{x}} - x = \lim_{y\to 0^+} \frac{e^y - 1}{y}$

or: $e^u = 1 + u + \frac{u^2}{2} + \ldots$

so $e^{\frac{1}{x}} = 1 + \frac{1}{x} + \frac{1}{2x^2} + \ldots$

Q: A plane passes over a ground station at altitude 13 km, climbing at an angle of 25° at $4\frac{km}{min}$. At what rate is the distance to the radar station changing 2 min later?



Pretend that the plane started on the ground at time 0.

~~[illegible]~~ Put origin of coordinate system at that point.

At time t, the plane is at distance vt from origin, so at height $vt\sin\theta$

distance $vt\cos\theta$

plane is over radar at time t_0 when $vt_0\sin\theta = H$

so $t_0 = \frac{H}{v\sin\theta}$

radar is at $x_0 = vt_0\cos\theta = \frac{\cos\theta}{\sin\theta}\cdot H$

So distance to radar is:

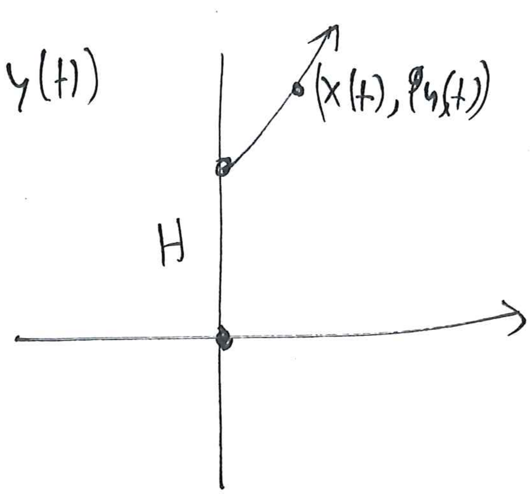
$$D^2 = \left(vt\cos\theta - \frac{\cos\theta}{\sin\theta}H\right)^2 + v^2t^2\sin^2\theta$$

now plug in $t = t_0 + 2$ to get D

diff to get $2D\cdot\frac{dD}{dt} = \ldots$

Alternative: say plane is at $(x(t), y(t))$

radar at $(0,0)$, $t=0$ over radar



$$D^2 = x^2 + y^2 \Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

but $\frac{dx}{dt} = v\cancel{\sin}\theta$, $\frac{dy}{dt} = v\sin\theta$

So

$$D\frac{dD}{dt} = xv\cos\theta + yv\sin\theta$$

$$x = vt\cos\theta \qquad y = H + vt\sin\theta$$

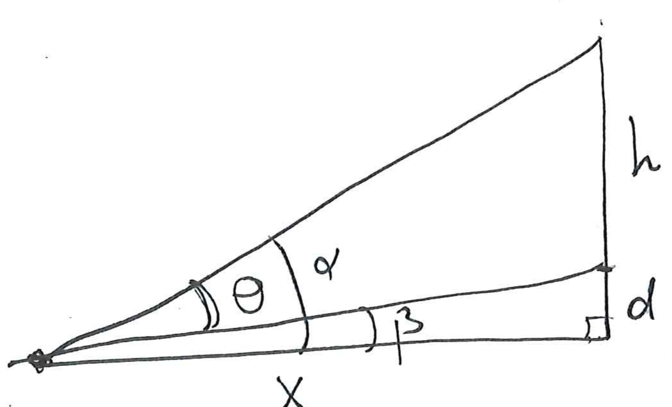
so $$D\frac{dD}{dt} = v^2t\cos^2\theta + (H+vt\sin\theta)v\sin\theta$$

plug in $V = 9\frac{km}{min}$ $t = 2min$, $\theta = 25°$, $H = 13km$

$$D = \sqrt{v^2t^2\cos^2\theta + (H+vt\sin\theta)^2}$$

Q: Optimization)

At what distance x does the object of length h subtend the largest angle?



← diagram

Let θ be the viewing angle. Let β be the angle subtended by base (of length d), α the angle subtended by whole object of length $d+h$

names

Then $\tan\beta = \frac{d}{x}$, $\tan\alpha = \frac{d+h}{x}$, $\theta = \alpha - \beta$ } relations

So $\theta = \arctan\frac{d+h}{x} - \arctan\frac{d}{x}$

Goal: maximize $\theta = \theta(x)$ on $(0,\infty)$

Always $\frac{d+b}{x} > \frac{d}{x} > 0$ so $\arctan\frac{d+h}{x} > \arctan\frac{d}{x}$ so $\theta > 0$

θ cts as a fcn of x.

$\lim_{x\to 0}\theta(x) = \lim_{x\to 0}\arctan\frac{d+h}{x} - \lim_{x\to 0}\arctan\frac{d}{x} = \frac{\pi}{2} - \frac{\pi}{2} = 0$

$\lim_{x\to\infty}\theta(x) = \lim_{x\to\infty}\arctan\left(\frac{d+h}{x}\right) + \lim_{x\to\infty}\arctan\frac{d}{x} = \arctan 0 - \arctan 0 = 0.$

So maximum of θ must occur between $(0,\infty)$ hence at a critical point.

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{d+h}{x}\right)^2}\cdot\left(-\frac{d+h}{x^2}\right) - \frac{1}{1+\left(\frac{d}{x}\right)^2}\cdot\left(-\frac{d}{x^2}\right)$$

$$= -\frac{d+h}{x^2+(d+h)^2} + \frac{d}{x^2+d^2} = \frac{d(x^2+(d+h)^2) - (d+h)(x^2+d^2)}{(x^2+(d+h)^2)(x^2+d^2)} =$$

$$= \frac{d(d+h)^2 - hx^2 - (d+h)d^2}{*} = \frac{d(d+h)h - hx^2}{*}$$

So $\frac{d\theta}{dx} > 0$ if $x < \sqrt{d(d+h)}$, $\frac{d\theta}{dx} < 0$ if $x > \sqrt{d(d+h)}$

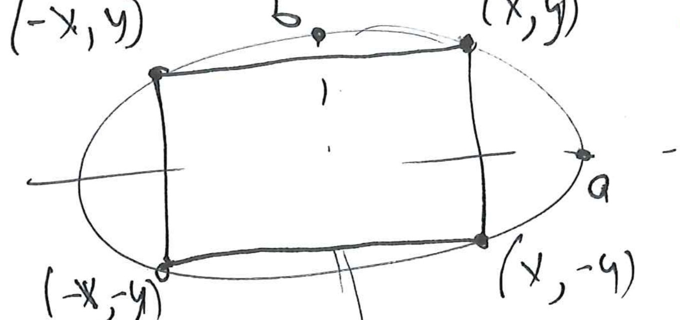
and $\frac{d\theta}{dx} = 0$ iff $x = \sqrt{d(d+h)}$, so the maximal angle is at $x = \sqrt{d(d+h)}$

Q: (Optimization)

Find the rectangle of largest area inscribed in an ellipse.

A: Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

[If a rectangle is inscribed in the ellipse then it is axis parallel]



← picture + names

Area of the rectangle is

$$A = 2x\cdot 2y = 4xy = 4bx\,\frac{y}{b} = 4bx\sqrt{1-\frac{x^2}{a^2}}$$

relations

$$= \frac{4b}{a}\,x\sqrt{a^2-x^2}$$

Calculus

So need to maximize $A(x)=\frac{4b}{a}x\sqrt{a^2-x^2}$ on $[0,a]$

$A(0)=0$, $A(a)=0$,

$$\frac{dA}{dx} = \frac{4b}{a}\left[\sqrt{a^2-x^2} + x\,\frac{-2x}{2\sqrt{a^2-x^2}}\right]$$

$$= \frac{4b}{a}\left[\frac{a^2-x^2}{\sqrt{a^2-x^2}} - \frac{x^2}{\sqrt{a^2-x^2}}\right] = \frac{4b\,(a^2-2x^2)}{a\sqrt{a^2-x^2}}$$

no singular pts in $(0,a)$, critical pt at $x=\frac{a}{\sqrt{2}}$ must be max since A vanishes at endpoints ~~So the largest~~

So the rectangle of largest area has one vertex at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ has sizes $\sqrt{2}\cdot a$, $\sqrt{2}\, b$

Q: IVT / MVT

(1) show that if f'' exists, and f has three roots, then f' has two roots, f'' has at least one

(2) Show that $2x^2 - 3 + \sin x + \cos x = 0$ has exactly two solutions

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(1) Say $f(a) = f(b) = f(c) = 0$ for $a < b < c$

By Rolle's thm (MVT when $f(a) = f(b) = 0$ $\frac{f(b) - f(a)}{b-a} = 0$) there are $d \in (a,b)$ $e \in (b,c)$

s.t. $f'(d) = 0$, $f'(e) = 0$. Applying Rolle's thm to f' on $[d,e]$ get $g \in (d,e)$ s.t. $(f')'(g) = 0$

(2) let $f(x) = 2x^2 - 3 + \sin x + \cos x$. Then f, f', f'' exist (f everywhere defined by formula), so f is continuous

$f(0) = -3 + 1 = -2 < 0$ $\quad f(10) = 197 + \sin 10 + \cos 10 \geq 195$

$f(-10) = 197 - \sin 10 + \cos 10 \geq 195$

By IVT, f has a root in $(-10,0)$, and a root in $(0,10)$

$\Rightarrow$ f has at least 2 roots

If f had 3 roots then by parts (1), f'' would have a root. But

$$f''(x) = \overset{4}{\cancel{2}} - \sin x - \cos x \geq 4 - 1 - 1 \geq 2 > 0$$

never vanishes. Thus f has at most 2 roots, so exactly two: the equation $f(x) = 0$ has exactly two solutions