Math 100, lecture 22, 30/11/2021

Last time: Anti-derivatives.

main idea: massaging expressions
+ arbitrary constant
+ boundary conditions

Today: Review

Overview (1) limits, continuity
(2) definition of derivative; diff rules.
(3) Applications of derivative:
related rates, optimization, curve sketching
linear, non-linear approximation
Taylor expansion
(4) Anti-derivatives

Q: MVT?

MVT is remainder estimate for constant approx:
If f is cts on [a,b], diff on (a,b) then there c between a,b s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c) \iff f(b) = f(a) + f'(c)(b-a)$$

Problem: show that the equation $\tan x = x+1$ has a solution

Solution: let $f(x) = \tan x - x$ (or $f(x) = \tan x - x - 1$)

want c s.t. $f(c) = 1$.   (want $f(c) = 0$)

① We'll try IVT, need: points a, b with $f(a) < 1$, $f(b) > 1$, f cts on $[a,b]$

② But $f(x)$ not cts everywhere: it blows up at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

③ can use the blowup to find a, b

$\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty$, $\lim_{x \to -\frac{\pi}{2}^+} x = -\frac{\pi}{2}$ so $\lim_{x \to -\frac{\pi}{2}^+} f(x) = -\infty$

so there is a close to $-\frac{\pi}{2}$ but $a > -\frac{\pi}{2}$, s.t. $f(a) < 1$

similarly

$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$, $\lim_{x \to \frac{\pi}{2}^-} x = \frac{\pi}{2}$ so $\lim_{x \to \frac{\pi}{2}^-} f(x) = \infty$,

thus there is b close to $\frac{\pi}{2}$ but $b < \frac{\pi}{2}$ s.t. $f(b) > 1$.

Since $[a,b] \subset (-\frac{\pi}{2}, \frac{\pi}{2})$, and f is defined by formula everywhere on $[a,b]$, f is cts there. By IVT have $a < c < b$ s.t. $f(c) = 1$

then $\tan c - c = 1$ so $\boxed{\tan c = c + 1}$.

Problem: Evaluate $\lim_{x \to 0} \frac{\cos x - e^{x^2}}{x^2}$.

Solution:

denominator vanishes at 0 to 2^{nd} order, so need at least 2^{nd} order expansion of numerator

(1) $\cos 0 = 1$, $(\cos')(0) = -\sin 0 = 0$, $(\cos'')(0) = -\cos 0 = -1$

so $\cos x \approx 1 - \frac{1}{2}x^2$ to 2^{nd} order.

(2) $e^0 = 1$, all derivatives of e^u are e^u, so all derivatives are 1 at 0.

$e^u \approx 1 + u + \frac{u^2}{2} + \ldots$

So $e^{x^2} \approx 1 + x^2 + \frac{x^4}{2} + \ldots \approx 1 + x^2$ to 2^{nd} order.

(changed variables to $u = x^2$; $x = 0 \Rightarrow u = 0$, expand about this point, plug in x^2 for u)

Putting together, $\cos x - e^{x^2} \approx \left(1 - \frac{x^2}{2}\right) - (1 + x^2) = -\frac{3}{2}x^2$ to 2^{nd} order.

So $\frac{\cos x - e^{x^2}}{x^2} \approx -\frac{3}{2}$ to 0^{th} order

So $\lim_{x \to 0} \frac{\cos x - e^{x^2}}{x^2} = -\frac{3}{2}$.

Taylor expansion of x^2 about 0? (degree ≥ 2)

The expansion is a polynomial, agree with x^2 for derivatives at 0.

Asides to 4th order,

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}, \quad e^{x^2} \approx 1 + x^2 + \frac{x^4}{2}$$

so $\cos x - e^{x^2} \approx -\frac{3}{2}x^2 - \frac{11}{24}x^4$ to 4th order

so $\frac{\cos x - e^{x^2}}{x^2} \approx -\frac{3}{2} - \frac{11}{24}x^2$ to 2nd order.

Question: remainder estimates for

$$\cos x - e^{x^2} \approx -\frac{3}{2}x^2$$

$-\frac{3}{2}x^2$ = both 2nd degree and third degree expansion

then $\cos x - e^{x^2} = -\frac{3}{2}x^2 + R_3(x)$

where $R_3(x) = \frac{f^{(4)}(c)}{4!}(x-0)^4$

$(e^{x^2})' = 2xe^{x^2}$, $(e^{x^2})'' = 2e^{x^2} + 4x^2e^{x^2} = (4x^2+2)e^{x^2}$

$(e^{x^2})''' = (8x^3 + 4x + 8x)e^{x^2} = (8x^3 + 12x)e^{x^2}$

$(e^{x^2})^{(4)} = (16x^4 + 24x^2 + 24x^2 + 12)e^{x^2} = (16x^4 + 48x^2 + 12)e^{x^2}$

so $\cos x - e^{x^2} = -\frac{3}{2}x^2 + \left[-\frac{1}{6}(4c^4 + 12c^2 + 3)e^{c^2} + \frac{\cos c}{24}\right]x^4$

for some c between 0, x.

E.g.: $x = \frac{1}{10}$ set

$$\cos\left(\tfrac{1}{10}\right) - e^{1/100} = -\frac{3}{2}\cdot\frac{1}{100} - \frac{(4c^4+12c^2+3)e^{c^2}}{60{,}000} + \frac{\cos c}{24{,}000}$$

$0 < c < \frac{1}{10}$

so error in approximating

$$\cos\left(\tfrac{1}{10}\right) - e^{1/100} \approx -\frac{3}{200}$$

$2^{100} > e$

is at most

$$-\frac{\cos c}{24{,}000} + \frac{\left(3 + \frac{12}{100} + \frac{4}{10{,}000}\right)e^{1/100}}{60{,}000} \leq \frac{3.2\cdot 2}{6000} \leq \frac{1}{6{,}000} + \frac{1}{24{,}000}$$

$$+ \frac{1}{24{,}000}$$

$$3 + \frac{12}{100} + \frac{4}{10{,}000} = 3.1204$$

If have $h(x) = f(g(x))$ expand about $x = a$.

(1) expand $g(x)$ about $x = a$

(2) expand $f(u)$ about $u = g(a)$

Plug in (1) into (2)

Problems

$$\lim_{x\to 0^+} (\sin x)^{\frac{1}{\log x}} = \lim_{x\to 0^+} e^{\log\left((\sin x)^{1/\log x}\right)} = e^{\lim_{x\to 0^+} \frac{\log \sin x}{\log x}}$$

↑ has form 0^0

↑ continuity of e^x

$$\overset{\text{l'Hôpital}}{=} e^{\lim_{x\to\infty} \frac{\cos x/\sin x}{1/x}} = e^{\lim_{x\to 0^+} \frac{\cos x}{\sin x/x}} = e^{(\lim_{x\to 0^+} \cos x)/(\lim_{x\to 0^+} \frac{\sin x}{x})} =$$

$$= e^{\cos 0/1} = e^{1/1} = e.$$

↑ $\lim_{x\to 0^+} \log \sin x = -\infty$ ($\sin x \to 0$ as $x\to 0^+$)

$\lim_{x\to 0^+} \log x = -\infty$

What if we don't recall that $\frac{\sin x}{x} \to 1$ as $x\to 0$?

can use l'Hôpital.

or: get

$$e^{\lim_{x\to 0^+} \frac{x\cos x}{\sin x}} \overset{\text{l'Hôpital}}{=} e^{\lim_{x\to 0} \frac{\cos x - x\sin x}{\cos x}} = e^{\frac{1-0}{1}} = e$$

↑ $\lim_{x\to 0} x\cos x = 0$

$\lim_{x\to 0} \sin x = 0$

Alternative: compute $\lim_{x\to 0^+} \log\left((\sin x)^{1/\log x}\right)$ first (set 1)

only exponentiate at the end

Alternative $\sin x \approx x - \frac{x^3}{6} + \ldots$

so $\sin x = x\left(1 - \frac{x^2}{6} + \ldots\right)$

so $\log \sin x \approx \log x + \log\left(1 - \frac{x^2}{6} + \ldots\right)$

$$\frac{\log \sin x}{\log x} \approx 1 + \frac{\log\left(1 - \frac{x^2}{6}\right)}{\log x} \quad x \to 0$$