17. Optimization (9/11/2021)

Goals.

(1) Problem solving
(2) Examples

Last Time. **Extrema of functions**

If f is cts on $[a,b]$ then f has min & max there, occuring at one of

(1) Critical points $f'(x_0) = 0$
(2) Singular points $f'(x_0)$ DNE
(3) end points a, b.

Aside: really need $[a,b]$: $f(x) = x+3$ does not have max on $(0,1)$.

How do we compute Taylor expansions?

Know: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$

$(-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)}{4}$

Can: plug into expansions

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\log(1-2x) = (-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - -$$

& also add, subtract, multiply –

to expand $e^x \cos x \log(1+x)$ to 5^{th} order.

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right)$$

$$\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)$$

multiply, discard terms above 5^{th} order

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expand $\sqrt{1+(x-2)^2}$ about $a=2$

solution: look at $f(u) = \sqrt{1+u}$ expand this about $u=0$

plug in $u = (x-2)^2$

Aside: $(1+u)^{1/2} = 1 + \frac{1}{1!}\frac{1}{2}u + \frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdot u^2$

$$+ \frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)u^3 + \frac{1}{4!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)u^4$$

$$f(x) \approx f(a) + f'(a)\cdot(x-a)$$

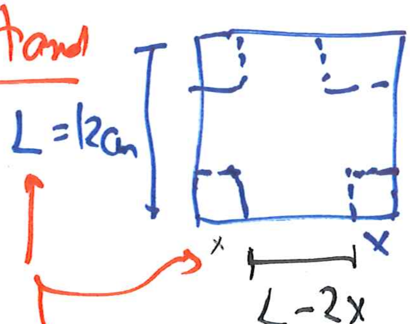
$$g(x) \approx g(a) + g'(a)\,(x-a)$$

$$f(x)\,g(x) \approx f(a)\,g(a) + \big(f(a)\,g'(a) + f'(a)\,g(a)\big)(x-a) + O\big((x-a)^2\big)$$

Today: Optimization

Example: we have a square of cardboard 12cm × 12cm want to cut corners off, fold rest into a box. what is the largest ~~box~~ volume box we can make?

(0) understand



cut squares of length x cm from corners, $0 \le x \le L/2$

let V = volume of resulting box.

(1) names

Then $V = (L-2x)^2 \cdot x$ (2) relations

(3) calculus V is cts & diff on $[0, L/2]$, $V'(x) = 2(L-2x)(-2)\cdot x + (L-2x)^2$

so $V'(x) = (L-2x)\big(-4x + L - 2x\big) = (L-2x)(L-6x)$

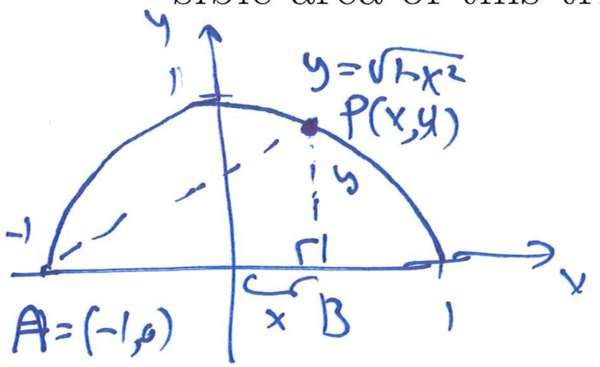
$\Rightarrow$ Critical pt at $x = L/6$, endpoints $0, L/2$

$V(0) = V\left(\frac{L}{2}\right) = 0$, $V(L/6) = (L - L/3)^2 \cdot L/6 = \frac{2}{27}L^3$

(4) endgame: Largest box we can make has volume $128\,cm^3$, obtained by cutting at $x = 2$ (side 8, height 2)

Math 100 – WORKSHEET 17
OPTIMIZATION

(1) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1-x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of this triangle?



Let $B=(x,0)$, $P=(x,y)$

Then the area is

$$f(x) = \frac{1}{2}\underbrace{(x-(-1))}_{\text{base}}\underbrace{y}_{\text{height}}$$

or $$f(x) = \frac{1}{2}(x+1)\sqrt{1-x^2}$$

f is cts on $[-1,1]$ $\quad f(\pm 1) = 0$

$$f'(x) = \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(x+1)\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = \frac{1}{2\sqrt{1-x^2}}\left(1-x^2+(x+1)(-x)\right)$$

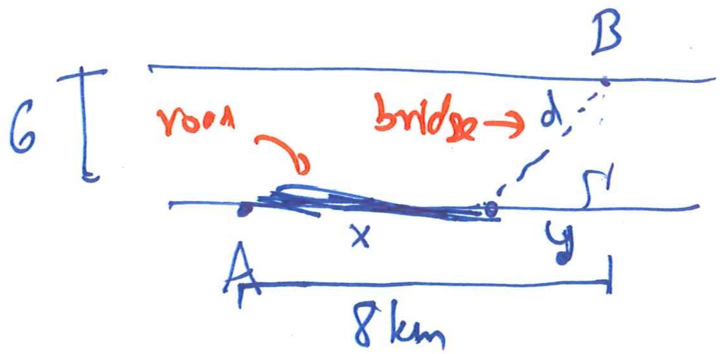
$$= \frac{1}{2\sqrt{1-x^2}}(x+1)(1-x-x) = \frac{(1+x)(1-2x)}{2\sqrt{1-x^2}}$$

vanishes only at $x=\frac{1}{2}$ in $(-1,1)$

so maximum is at $x=\frac{1}{2}$, $f\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$

Date: 9/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?



say road has length x
bridge length d

$d=\sqrt{6^2+(8-x)^2}$

So cost is $C(x)=20x+40\sqrt{6^2+(8-x)^2}$

or: $C(y)=20(8-y)+40\sqrt{6^2+y^2}$

Clearly minimum cost is for $0\le x\le 8$.

$C(0)=40\sqrt{100}=400$, $C(8)=20\cdot 8+40\cdot 6=400$

in the middle C is cts, diff for x:

$$C'(x)=20+40\cdot\frac{2(x-8)}{2\sqrt{6^2+(x-8)^2}},$$ solve $C'(x)=0$

$-20\sqrt{6^2+(x-8)^2}=40\cdot(x-8)$ squaring,

$6^2+(x-8)^2=4(x-8)^2$ so $3(x-8)^2=36$ so $x=8-\sqrt{12}$

So the cheapest way to construct a bridge is to build a road of length $8-\sqrt{12}$ km, and then a bridge of length $\sqrt{48} = 4\sqrt{3}$ km

(cheapest because the cost is

$$40\sqrt{48} \quad 40\sqrt{3}$$

$$20\left(8-\sqrt{12}\right) + 40\sqrt{6^2 + \sqrt{12}^2} = 160 + \cancel{\phantom{40\sqrt{48}}}$$

$$= 160 + 120\sqrt{3} < 160 + 120 \cdot 2 = C(0) = C(8))$$

(or: $C'(0) = 40 \cdot \dfrac{-8}{\sqrt{6^2+8^2}} < 0$, so C is decreasing at 0, so minimum not at 0, so $8-\sqrt{12}$ must be the minimum.)