7. Computing derivatives (5/10/2021)

Goals.

(1) The product and quotient rules
(2) The tangent line

Last Time.

Definition: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ (if limit exists)

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Uses: (1) compute derivatives directly.
(2) recognize limits as derivatives

Computing derivatives using rules:

(1) Linearity: $(af+bg)' = af' + bg'$

(2) power laws: $(x^a)' = a x^{a-1}$.

~~The~~ line tangent to $y=f(x)$ at $(a, f(a))$ is $y = f'(a)(x-a) + f(a)$

Math 100 – WORKSHEET 7
DIFFERENTIATION RULES

1. The product and quotient rules

(1) Differentiate

(a) $f(x) = 6x^\pi + 2x^e - x^{7/2}$

$$\frac{df}{dx} = 6\pi x^{\pi-1} + 2e\, x^{e-1} - \frac{7}{2} x^{5/2}$$

(b) (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

$$\frac{dg}{dx} = \frac{d(x^2)}{dx} \cdot e^x + x^2 \frac{d(e^x)}{dx} = 2x \cdot e^x + x^2 e^x = (2x + x^2)e^x.$$

$$\boxed{\neq \frac{d(x^2)}{dx} \cdot \frac{d(e^x)}{dx}}$$

Date: 5/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Why are these rules true?

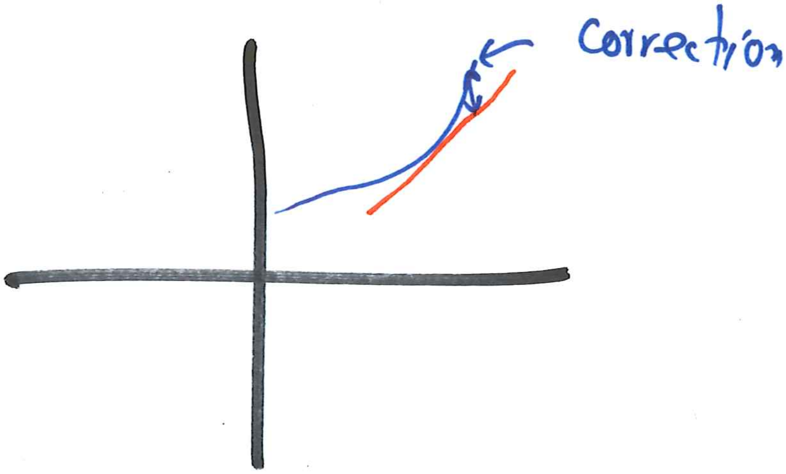
Answer: because of the linear approximation

Recall: $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$\iff \lim_{x \to a} \left[\dfrac{f(x) - f(a)}{x - a} - f'(a) \right] = 0$

$\iff \lim_{x \to a} \dfrac{f(x) - [f(a) + f'(a)(x-a)]}{x - a} = 0$

$\Rightarrow \quad f(x) = f(a) + f'(a)(x-a) + \underbrace{\text{small correction}}_{R = \text{"remainder"}}$



small: $\dfrac{R(x)}{x-a} \xrightarrow[x \to a]{} 0$

Suppose
$$f(x) = f(a) + f'(a)(x-a) + \text{small}$$
$$g(x) = g(a) + g'(a)(x-a) + \text{small}$$
add

$(f+g)(x) = (f+g)(a) + (f'(a) + g'(a))(x-a) + \text{small}$

$(fg)(x) = (fg)(a) + f(a)g'(a)(x-a) + f'(a)g(a)(x-a) + f'(a)g'(a)(x-a)^2 + \text{small}$

$= (fg)(a) + (f(a)g'(a) + f'(a)g(a))(x-a) + \text{small}$

(c) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$$h'(x) = \frac{(x^2+3)' \cdot (2x-1) - (x^2+3)(2x-1)'}{(2x-1)^2} = \frac{2x(2x-1) - (x^2+3) \cdot 2}{(2x-1)^2} =$$

$$= \frac{2x^2 - 2x - 6}{(2x-1)^2}$$

(d) $\frac{x^2+A}{\sqrt{x}}$ $= x^{3/2} + Ax^{-\frac{1}{2}}$ so $\left(\frac{x^2+A}{\sqrt{x}}\right)' = \frac{3}{2}x^{1/2} - \frac{A}{2}x^{-3/2}$

or $\left(\frac{x^2+A}{\sqrt{x}}\right)' = \frac{2x \cdot \sqrt{x} - (x^2+A) \cdot \frac{1}{2\sqrt{x}}}{\sqrt{x}^2} =$

$$= 2\sqrt{x} - \frac{1}{2}\sqrt{x} - \frac{A}{2x^{3/2}} = \frac{3}{2}x^{1/2} - \frac{A}{2}x^{-3/2}$$

(2) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

Story:

(1) compute $f'(x)$

(2) plug 4 into $f'(x)$

$\Downarrow$

(3) get equation for A

$$f'(x) = \frac{(\sqrt{x}+A) - x\frac{1}{2\sqrt{x}}}{(\sqrt{x}+A)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2}$$

$$\text{So } f'(4) = \frac{\frac{1}{2}\sqrt{4}+A}{(\sqrt{4}+A)^2} = \frac{1+A}{(2+A)^2}$$

$$\text{So } \frac{1+A}{(2+A)^2} = \frac{3}{16} \qquad \text{So } \frac{3}{16}(2+A)^2 = 1+A$$

$$\Leftrightarrow \boxed{\frac{3}{16}A^2 - \frac{1}{4}A - \frac{1}{4} = 0}$$

(3) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$. Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

$$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 6 + 4 = 10$$

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{6-4}{2^2} = \frac{1}{2}$$

2. The tangent line

(1) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

$f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(4) = \frac{1}{4}$ (slope)

y-value

so line is $y = \frac{1}{4}(x-4) + 2$

$\Leftrightarrow y = \frac{1}{4}x + 1 \Leftrightarrow 4y - x = 4$

(2) Let $f(x) = \frac{g(x)}{x}$, where $g(x)$ is differentiable at $x = 1$. The line $y = 2x - 1$ is tangent to the graph $y = f(x)$ at $x = 1$. Find $g(1)$ and $g'(1)$.

$2 \cdot 1 - 1$

The line has slope 2, & the point of tangency is $(1, 1)$

so $f'(1) = 2$, $f(1) = 1$

So $1 = f(1) = \frac{g(1)}{1} = g(1)$ so $\boxed{g(1) = 1}$

$\frac{df}{dx} = f' = \frac{g'(x) \cdot x - g(x)}{x^2}$ so $2 = f'(1) = \frac{g'(1) \cdot 1 - g(1)}{1^2}$

so $\boxed{g'(1) = 2 + g(1) = 3}$

Or: $g(x) = x f(x)$ so $g(1) = 1 \cdot f(1) = 1$

$g'(1) = f(1) + \cancel{x} \cdot f'(1) = 1 + 2 = 3$

(3) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

tangent = slope 4, point (1, 6) on graph

so for each possible f, check if $f(1) = 6$

$f'(1) = 4$.

(4) Find the lines of slope 3 tangent the curve $y = x^3 + 4x^2 - 8x + 3$.

(don't know where line is tangent)

Let a be the point of tangency, then slope at a is 3,

i.e. $\boxed{3a^2 + 8a - 8 = 3}$

give unknown quantity a name