Lior Silberman's Math 312: Problem Set 1

The factorial function and binomial coefficients

Recall that the factorial function is defined by $0! = 1$ and for $n \geq 0$ by $(n+1)! = (n+1) \cdot n!$. The *binomial coefficients* are defined for $0 \leq k \leq n$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n \geq 0$ we set $\binom{n}{k} = 0$ (for example, $\binom{4}{2} = 6$ while $\binom{2}{4} = 0$).

1. For $0 \leq k < n$ show that $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$ by a direct calculation.
   OPT: show that this holds even if $k \geq n$.

2. For $n \geq 0$ show that $\sum_{j=0}^{n} \binom{j}{0} = \binom{n+1}{1}$.
   Hint: once you unwind the definitions of both sides this is not hard.

Induction

Use mathematical induction to prove the following assertions:

3. Among every three consecutive positive integers there is one that is divisible by 3.

4. Fix $k \geq 0$ and show by induction on n that for all $n \geq 0$, $\sum_{j=0}^{n} \binom{j}{k} = \binom{n+1}{k+1}$.
   Hint: For the induction step use problem 1.

5. (Summation formulas) The case $k = 1$ of problem 4 reads: $\sum_{j=0}^{n} j = \binom{n+1}{2} = \frac{n(n+1)}{2}$. In this problem we will establish similar formulas for summing squares and cubes of integers (you may recall these formulas from your integral calculus course). Please express the formulas in the same form: a product of terms linear in n divided by an integer.
   (a) Show that $j^2 = 2\binom{j}{2} + \binom{j}{1}$. This means that $\sum_{j=0}^{n} j^2 = 2\sum_{j=0}^{n} \binom{j}{2} + \sum_{j=0}^{n} \binom{j}{1}$ (why?). Use problem 4 to establish a formula for $\sum_{j=0}^{n} j^2$.
   (b) Express j^3 as a combination of $\binom{j}{3}$, $\binom{j}{2}$, $\binom{j}{1}$ and use problem 4 to prove a formula for $\sum_{j=0}^{n} j^3$.
   RMK You can check your formlas (but not your proofs) on the reverse page.

6. (Well-ordering) Use the well-ordering principle to show that every amount of money payable with only dimes (10¢) and quarters (25¢) is divisible by 5¢.

Factorials and primes

7. For $2 \leq j \leq n$, show that $n! + j$ is not a prime number. Conclude that there are arbitrarily long intervals containing no prime numbers.

8. For which prime numbers p is $p+1$ also prime?

REMARK. It is believed (the "twin prime conjecture") that there are infinitely many primes p for which $p+2$ is also prime. It is known (Polymath 8's refinement of Zhang's Theorem) that there is some k, $2 \leq k \leq 246$ such that $p, p+k$ are both prime infinitely often.

Hint for problem 5:

$$\begin{aligned}
\sum_{j=0}^{n} 1 &= n+1 \\
\sum_{j=0}^{n} j &= \frac{n(n+1)}{2} \\
\sum_{j=0}^{n} j^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{j=0}^{n} j^3 &= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

Supplementary problem: the Peano Axioms

Our axiom system for the integers is redundant. In this supplementary problem (**NOT FOR SUBMISSION**) we explore a bare-bones axiom system which is sufficient.

DEF The *natural numbers* are a triple $(\mathbb{N}, 0, S)$ where:

(0) $\mathbb{N}$ is a set, $0 \in N$, and $S \colon \mathbb{N} \to \mathbb{N}$is a function ("successor")

(1) ("0 is not the successor of any integer") $0 \neq Sn$ for all $n \in \mathbb{N}$

(2) For any $n, m \in \mathbb{N}$, if $Sn = Sm$ then $n = m$

(3) (Induction) $\mathbb{N}$ is the only inductive set, where $A \subset \mathbb{N}$ is "inductive" if $0 \in A$ and if $Sn \in A$ whenever $n \in A$.

Notation $1 = S0$, $2 = S1 = SS0$, $3 = S2 = SS1 = SSS0$ and so on.

A. Let $A = \{n \in \mathbb{N} \mid n = 0 \text{ or } n = Sm \text{ for some } m \in\}$. Show that A is inductive and conclude that every non-zero natural number is a successor.

B. (Addition) Define a function $+ \colon \mathbb{N} \to \mathbb{N}$ as follows: for all $n, m \in \mathbb{N}$: (a) $n + 0 = n$ (b) $n + Sm = S(n+m)$. [informally, we define $n + (m+1)$ by $(n+m)+1$.

(a) Show that $2+1 = 3$ and that $1+2 = 3$ (hint: $2+1 = 2+S0$)

(b) Show that function $+$ is determined by the conditions: suppose $+'$ also satisfies the definition, any for any $n \in \mathbb{N}$ set $A = \{m \mid n+m = n+' m\}$. Then A is inductive so $n+m = n'+m$ for all n, m.

(c) (Associative law) By induction on k, show that for any fixed $n, m \in \mathbb{N}$, $(n+m)+k = n+(m+k)$ for all $k \in \mathbb{N}$.

(d) (Zero) By induction on n, show that $0+n = n+0 = n$ holds for all n.

(e) (Commutative law) By induction on m, show that the statement "for all n, $m+n = n+m$" holds for any m.

(f) (Cancellative law) By induction on k, show that the statement "for all n, m if $n+k = m+k$ then $n = m$" holds for any k.

C. (Order) For $m,n \in \mathbb{N}$ say $m \leq n$ if there is $k \in \mathbb{N}$ such that $n = m+k$.
   (a) Let $A = \{n \in \mathbb{N} \mid n = 0 \text{ or } n \geq 1\}$. Show that A is inductive, and conclude that $A = \mathbb{N}$.
   (b) Given $m \in \mathbb{N}$ let $A = \{n \in \mathbb{N} \mid n \leq m \text{ or m≤n}\}$. Show that A is inductive, and conclude that for any m,n at least one of $m \leq n, n \leq m$ holds.
   (c) Suppose that $k+l = 0$. Show that $k = l = 0$ (hint: if k or l is non-zero is a successor, and this makes $k+l$ a successor, which contradicts problem A).
   (d) Suppose $m \leq n$ and $n \leq m$. Show that $n = m$.
   (e) Let $m,n,k \in \mathbb{N}$. Show that $m \leq n$ holds iff $m+k \leq n+k$ holds.

D. (Multiplication) Define a function $\cdot \colon \mathbb{N} \to \mathbb{N}$ as follows: for all $n,m \in \mathbb{N}$: (a) $n \cdot 0 = 0$ (b) $n \cdot Sm = n \cdot m + n$. [informally, we define $n \cdot (m+1)$ by $(n \cdot m) + m$.
   (a) Show that $n \cdot 1 = n$ for all n and that $2 \cdot 2 = 4$.
   (b) Show that function $\cdot$ is determined by the conditions.
   (c) (Distributive law) For any n,m show by induction on k that $n \cdot (m+k) = n \cdot m + n \cdot k$.
   (c) (Associative law) For any n,m show by induction on k that $(n \cdot m) \cdot k = n \cdot (m \cdot k)$.
   (d) By induction on n, show that $0 \cdot n = 0 = n \cdot 0$ holds for all n.
   (e) (Commutative law) By induction on m, show that the statement "for all n, $m \cdot n = n \cdot m$" holds for any m.
   (f) (Cancellative law) By induction on $k \geq 1$, show that the statement "for all n,m if $n \cdot k = m \cdot k$ then $n = m$" holds for any k.

E. Show that for any $k \geq 1$ and any $n,m \in \mathbb{N}$, $n \leq m$ iff $n \cdot k \leq m \cdot k$.

F. (Well-ordering) Let $P(n)$ be the statement "Suppose $A \subset \mathbb{N}$ contains an element k such that $k \leq n$. Then A has a least element".
   (a) Show that $P(0)$ holds.
   (b) Show that if $P(n)$ holds then $P(n+1)$ holds.
   (c) Conclude that $\mathbb{N}$ is well-ordered: every non-empty $A \subset \mathbb{N}$ has a least element.