Math 223: Problem Set 10 (due 16/11/2012)

Practice problems

Section 5.1: all problems are suitable
Section 5.2: all problems are suitable

PRAC Let $T, T' \in \mathrm{End}(V)$ be similar. Show that $p_T(x) = p_{T'}(x)$. (Hint: show that $x\mathrm{Id} - T$, $x\mathrm{Id} - T'$ are similar)

Calculation

1. Find the characteristic polynomial of the following matrices.

(a) $\begin{pmatrix} 5 & 7 \\ -3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} \pi & e \\ \sqrt{7} & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}$.

2. For each of the following matrices find its spectrum and a basis for each eigenspace.

(a) $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ (b) $\frac{1}{3}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Projections

Fix a vector space V.

3. Let $T \in \mathrm{End}(V)$, $p \in \mathbb{R}[x]$ and suppose that $p(T) = 0$. Show that $p(\lambda) = 0$ for all eigenvalues λ of V.

4. Let $P \in \mathrm{End}(V)$ satisfy $P^2 = P$. Such maps are called *projections*.
   (a) Apply problem 3 to show that $\mathrm{Spec}(P) \subset \{0, 1\}$.
   (b) Show that $(I - P)$ is a projection as well.
   (c) Show $V_1 = \mathrm{Im}\, P$.
   (*d) Note that $V_0 = \mathrm{Ker}\, P$ by definition. Show that $V_0 = \mathrm{Im}(I - P)$ and conclude that $V = V_0 \oplus V_1$.
   (*e) Converse: let $V_0, V_1 \subset V$ be arbitrary subspaces such that $V = V_0 \oplus V_1$. Show that there exists a unique $P \in \mathrm{End}(V)$ such that $P(\underline{v}_0) = \underline{0}$, $P(\underline{v}_1) = \underline{v}_1$ for all $\underline{v}_i \in V_i$, and that this P is a projection.
   DEF This P is called the *projection onto* V_1 *along* V_0.
   (f) Let $V_0 = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ $V_1 = \mathrm{Span}\left\{ \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ so that $\mathbb{R}^3 = V_0 \oplus V_1$ [check for yourself]. Let P be the projection onto V_1 along V_0. Find the matrix of P with respect to the *standard basis* of $\mathbb{R}^3$.

The Quantum Harmonic Oscillator, I

PRAC In physics a "parity operator" is a map $R \in \mathrm{End}(V)$ such that $R^2 = \mathrm{Id}_V$.

RMK This is problem 5, but it is not for submission.

(a) Show that $\pm \mathrm{Id}_V$ are (uninteresting) parity operators.

(b) For parts (b)-(d) fix a parity operator R. Show that its eigenvalues are in $\{\pm 1\}$ and let $V_\pm$ be the corresponding eigenspaces.

(c) Show that $\frac{I+R}{2}$, $\frac{I-R}{2}$ are the projections onto V_+, V_- along the other subspace, respectively.

(d) Conclude that $V = V_+ \oplus V_-$ and hence that every parity operator is diagonalizable.

(e) Let X be a set and let $\tau \colon X \to X$ be an *involution*: a map such that $\tau^2 = \mathrm{Id}_X$. Let $R_\tau \in \mathrm{End}(\mathbb{R}^X)$ be the map $f \mapsto f \circ \tau$. Show that P_τ is a parity operator.

(f) Let $X = \mathbb{R}$, $\tau(x) = -x$. Explain how (b)-(e) relate to the concepts of *odd* and *even* functions.

6. Let $V = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x] \right\}$ and for $n \geq 1$ let $V_n = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x]^{<n} \right\} \subset V$. Let $H \in C^\infty(\mathbb{R})$ be the operator ("quantum Hamiltonian") $H = -D^2 + M_{x^2}$. In other words we have $Hf = -\frac{d^2 f}{dx^2} + x^2 f$.

PRAC Show that $V_n \subset V$ are subspaces of $C^\infty(\mathbb{R})$, the space of infinitely differentiable functions.

(a) Show that $HV \subset V$ and $HV_n \subset V_n$.

(b) Let $H_n = H \restriction_{V_n} \in \mathrm{End}(V_n)$ be the restriction of H to V_n. Show that H_n has an upper-triangular basis with respect to an appropriate basis of V_n and determine its eigenvalues.

(c) Show that H_n is diagonable.

(d) Show that $HR = RH$ for the parity operator of 5(f).

(*e) Show that every eigenfunction of H_n is either even or odd. Which is which?

(f) Show that $V = \left\{ p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x] \right\}$ has a basis of eigenfunctions of H, and that each eigenfunction is either even or odd.

Supplementary problem: Nilpotent operators

A Let $N \in \mathrm{End}(V)$.

(a) Define subspaces $W_k \subset V$ by $W_0 = V$ and $W_{k+1} = NW_k$. Show that $W_k = \mathrm{Im}(N^k)$.

(b) Suppose that $W_{k+1} \subsetneq W_k$ for $0 \leq k \leq K-1$. Show that $\dim V \geq K$.

(c) Show that either the sequence $\{W_k\}_{k=0}^\infty$ reaches zero after at most $\dim V$ steps or there is a non-zero subspace $W \subset V$ such that $NW = W$.

(d) Suppose that $N^k = 0$ for some large k. Show that $N^n = 0$ where $n = \dim V$.

DEF N such that $N^k = 0$ is called *nilpotent*

(e) Find the spectrum of a nilpotent operator.