Lior Silberman's Math 223: Practice Problem Set 8

Practice problems

Section 4.1, Problems 1-8.
Section 4.2, Problems 1-23 (don't do all of them!)

1. Let V be a two-dimensional vector space, A an area form on V. Let $T \in \mathrm{End}(V)$ be linear. We know that the function $A'(\underline{u},\underline{v}) = A(T\underline{u},T\underline{v})$ is also an area form, and in fact that there is c such that for all $\underline{u},\underline{v} \in V$, $A'(\underline{u},\underline{v}) = cA(\underline{u},\underline{v})$. We then defined $c = \det T$. Let $\{\underline{v}_1,\underline{v}_2\}$ be a basis of V and let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be the matrix of T with respect to this basis. Calculate $A'(\underline{v}_1,\underline{v}_2)$ in terms of $A(\underline{v}_1,\underline{v}_2)$ and conclude that $\det T = \det B$ (use the definition of $\det B$ as in the textbook).

2. (Transpose 1) For a matrix $A \in M_{n,m}(\mathbb{R})$ the *transpose* of A is the matrix $A^t \in M_{m,n}(\mathbb{R})$ such that $(A^t)_{ij} = A_{ji}$.
   (a) The map $A \mapsto A^t$ is a linear map, and $(A^t)^t = A$.
   (b) Suppose that the product AB makes sense. Then $(AB)^t = B^tA^t$.

3. (Elementary matrices)
   (a) Check that if $i \neq j$ then $\det(I_n + cE^{ij}) = 1$.
   (b) Show that $\det \mathrm{diag}(a_1,\ldots,a_n) = \prod_{i=1}^n a_i = a_1a_2\cdots a_n$
   (c) Conclude that if E is one of the matrices from (a),(b) them $\det(A^t) = \det A$, where $(A^t)_{ij} = A_{ji}$.

4. (Transpose 2) Let $A \in M_n(\mathbb{R})$. Show that $\det A^t = \det A$.
   Hint: prove the theorem directly for matrices in reduced row echelon form, and then use the structure theorem and problem 3 for the general case.

5. (Vandermonde I) Calculate the following determinants: $V_2(x_1,x_2) = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}$, $V_3(x_1,x_2,x_3) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$.
   PRAC Can you guess a formula for $V_n(x_1,\ldots,x_n)$ the determinat of the matrix A such that $A_{ij} = x_i^{j-1}$?

Challenge problem – the Fifteen puzzle

6. The "fifteen puzzle" is played on a $n \times n$ grid. The puzzle consists of $n^2 - 1$ sliders, labelled with the numbers between 1 through $n^2 - 1$, placed on distinct grid points, leaving one grid point empty. We will call such a placement a *configuation* of the puzzle. A *legal move* consists of sliding one of the sliders vertically or horizontally into the empty position. For the purposes of a mathematical description we will replace the empty position with an additional slider marked "n^2", so that a configuation consists of a matrix $C \in M_n(\mathbb{R})$ with the entries being $1,2,3,\cdots,n^2$ in some order, and legal moves consists of exchanging the token marked "n^2" with one of its neighbours.
   DEF To go through the grid points in "natural order" means to go through the first row in order left-to-right, then the second row left-to-right and so on. We say a grid position occurs "later" than another if it will be checked later when going through the grid in order. Define the *number of crossings* of a configuation to be the number of pairs of grid points such that the number written in the later position of the two is smaller than the number written in the earlier one. Now define the *parity* $\varepsilon(C)$ of a configuration to be $+1$ if there is an even number of crossings, -1 if there is an odd one. Define the *total parity* to be the number $\delta(C) = \varepsilon(C) \times (-1)^{i+j}$ where (i,j) are the coordinates of the position marked n^2.

EXAMPLE ($n=3$) Let $C = \begin{bmatrix} 2 & 1 & 5 \\ 9 & 8 & 3 \\ 4 & 6 & 7 \end{bmatrix}$. Then the legal moves are to exchange the 9 with the $2,8$ or 4, the crossings are (in terms of the numbers written in the grid points, not in term of positions) $2\to 1$, $5\to 3$, $5\to 4$, $9\to 8$, $9\to 3$, $9\to 4$, $9\to 6$, $9\to 7$, $8\to 3$, $8\to 4$, $8\to 6$, $8\to 7$, the parity is $(-1)^{12}=1$ and the total parity is $(-1)^{10}(-1)^{2+1}=-1$ since the 9 is in position $2,1$.

(**a) Let C, C' be two positions connected by a single legal move. Show that $\varepsilon(C) = -\varepsilon(C')$ and that $\delta(C) = \delta(C')$.

(b) Let C, C' be two positions such that we can go from C to C' by $m \geq 0$ legal moves. Show that $\delta(C) = \delta(C')$.

(c) (Negative solution to the Fifteen Puzzle) Show that there is no sequence of legal moves that starts in the configuation $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 15 & 14 & E \end{bmatrix}$ and ends in the configuation $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & E \end{bmatrix}$. Here we denoted the empty position E rather than 16.

Supplement 1: Complex numbers

A. Let $\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a,b \in \mathbb{R} \right\} \subset M_2(\mathbb{R})$. We will denote elements of $\mathbb{C}$ by lower-case letters like z, w.

(a) Show that $\mathbb{C}$ is a subspace of $M_2(\mathbb{R})$. Conclude, in particular, that addition in $\mathbb{C}$ satisfies all the usual axioms.

(b) Show that $\mathbb{C}$ is closed under multiplication of matrices, that $I_2 \in \mathbb{C}$ and that $zw = wz$ for any $z, w \in \mathbb{C}$. It follows that multiplication in $\mathbb{C}$ is associative, commutative, has an identity, and is distributive over addition.

(c) Use PS5 problem 3 to show that every non-zero $z \in \mathbb{C}$ is invertible and derive a formula for the inverse.

DEF A set equipped with an addition and a multiplication operations which are commutative, associative, and have neutral elements, satisfying the distributive law and such tha every elemenent has an additive inverse, and every non-zero element has a multiplicative inverse, is called a *field*.

RMK The field $\mathbb{C}$ constructed above contains a copy of $\mathbb{R}$ – indeed by PS7 problem 3 (practice part) the identification $a \leftrightarrow \begin{pmatrix} a & \\ & a \end{pmatrix}$ respects addition and multiplication of real numbers; we do this from now on. [In fact, we already agreed to identify the number a with the linear map of multiplication by a].

(d) Let $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}$. Show that $i^2 = -1$ (note that $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) and that that every elememt of $\mathbb{C}$ can be uniquely written in the form $a + bi$ for some $a, b \in \mathbb{R}$ (hint: your answer should use the word "basis")

DEF From now on if asked to calculate a complex number write it in the form $a+bi$. Do NOT use the cumbersome specific realization of parts (a)-(d).

RMK Really try to forget the specific construction of parts (a)-(d) and only work in terms of the basis $\{1, i\}$. In particular, note that $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$ – you showed this for (b), but it also follows from the applying the distributive law and other laws of arithmetic and at some point using $i^2 = -1$.

(e) Calculate $(1+2i)+(3+7i)$, $(1+2i)\cdot(3+7i)$, $\frac{7+3i}{1+2i}$ (hint: division means multiplication by the inverse!)

EXAMPLE $(5-2i)\cdot(1+i) = 5\cdot(1+i)+(-2i)(1+i) = 5+5i-2i-2i\cdot i = 5+3i-2\cdot(-1) = 7+3i$.

B. (Inverting complex numbers using the norm)

DEF The *complex conjugate* of $z \in \mathbb{C}$ is the number $\bar{z}$ represented by the matrix z^t.

(a) Use problem 3 to show $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{zw} = \overline{zw}$. Also check that $\overline{a+bi} = a - bi$ and use this to give an alternate proof of the claims.

(b) Show that $z\bar{z}$ is a non-negative real for all $z \in \mathbb{C}$ (again we identify $a \in \mathbb{R}$ with the matrix aI_2), and that $z\bar{z} = 0$ iff $z = 0$. Conclude $z \neq 0$ then $z \cdot \frac{\bar{z}}{z\bar{z}} = 1$, a variant of the proof of A(c).

DEF The *norm* of $z\bar{z}$ is defined to be $|z| \stackrel{\text{def}}{=} \sqrt{z\bar{z}}$.

(c) Show that $|zw| = |z|\,|w|$. (Hint: this is easy using part (a) of this problem).

(d) Show that $\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$.

C. (Linear algebra over the complex numbers)

DEF A *complex vector space* is a triple $(V, +, \cdot)$ satisfying the usual axioms except that multiplication is by complex rather than real numbers.

DEF $\mathbb{C}^X$ is the space of C-valued functions on the set X. This is a complex vector space under pointwise operations (review the definition of $\mathbb{R}^X$). In particular, $\mathbb{C}^n$ is the space of n-tuples.

FACT Everything we proved about real vector spaces is true for complex vector spaces. For example, the standard basis $\{\underline{e}_k\}_{k=1}^n \subset \mathbb{C}^n$ is still a basis. We use $\dim_{\mathbb{C}} V$ to denote the dimension of a complex vector space, and when needed $\dim_{\mathbb{R}} V$ to denote the dimension of a real vector space.

(a) In the vector space $\mathbb{C}^2$ calculate $(1+2i) \cdot \begin{pmatrix} i \\ 3-7i \end{pmatrix}$. Show that $\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$ form a basis for $\mathbb{C}^2$.

(b) Show that $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix} \right\} \subset \mathbb{C}^2$ are linearly independent *over* $\mathbb{R}$ [that is: if a linear combination with real coefficients is zero, then the coefficients are zero].

RMK Since $\begin{pmatrix} a+bi \\ c+di \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} i \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} + d\begin{pmatrix} 0 \\ i \end{pmatrix}$ this set is also spanning,

(c) Solve the following system of linear equations over C:

$$\begin{cases} 5x + iy + (1+i)z & = 1 \\ 2y + iz & = 2 \\ -ix + (3-i)y & = i \end{cases}$$

More Supplementary problems

D. (Inefficiency of minor expansion) Suppose that the "minor expansion along first row" algorithm for evaluating determinants requires T_n multiplications to evaluate an $n \times n$ determinant.

(a) Show that $T_1 = 0$ and that $T_{n+1} = (n+1)(T_n + 1)$.

(b) Show that for $n \geq 2$ one has $T_n = n! \left(\sum_{j=1}^{n-1} \frac{1}{j!} \right)$

(c) Conclude that $n! \leq T_n \leq e \cdot n!$ for all $n \geq 2$.

E. Let X be a set. A *permutation* of X is a function $\sigma \colon X \to X$ which injective and surjective. The set of permutations of X is denoted S_X.

(a) Which of the following are permutations: (i) $\sigma(n) = n+1$ on $\mathbb{N}$; (ii) $\sigma(n) = n+1$ on $\mathbb{Z}$; (iii) $\sigma(n) = 2n$ on $\mathbb{Z}$; (iv) $\sigma(n) = 2n$ on $\mathbb{Q}$?

(b) (Group property) Suppose that $\sigma, \tau \in S_X$. Show $\mathrm{Id}_X, \sigma \circ \tau, \sigma^{-1} \in S_X$,

DEF When $X = \{1, 2, \cdots, n\}$we usually write S_n rather than $S_{\{1,\cdots,n\}}$, and write individual elements via their graphs like so: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$ for the map such that $\sigma(1) = 4$, $\sigma(2) = 1$, $\sigma(3) = 3$, $\sigma(4) = 2$.

(c) Calculate $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 3 & 2\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 3 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 3 & 2\end{pmatrix}\circ\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 4 & 1\end{pmatrix}$ (hint: plug in $1,2,3,4$ to the funciton on the right of the $\circ$, and then the output of that to the function on the left).

DEF Define the *crossing number* of $\sigma \in S_n$ to be the number $c(\sigma) \stackrel{\text{def}}{=} \{(i,j) \mid 1 \le i < j \le n, \sigma(i) > \sigma(j)\}$, and the *parity* (or *sign*) of σ to be the number $(-1)^{\sigma} \stackrel{\text{def}}{=} (-1)^{c(\sigma)}$

(d) Calculate the crossing number and parity of the permutations appearing in (c).