Lior Silberman's Set Theory: Problem set 3 on the integers

1. Prove by induction the laws of arithmetic in ω (the commutative and associative laws for addition and multiplication, the distributive law for multiplication over addition, the cancellative laws for addition and multiplication, the laws for adding constant to inequalities and of multiplying inequalities by constants).
2. Prove by induction the pigeon-hole principle: for any $n \in \omega$ any $f\colon n \to n$ is injective iff it is surjective.
3. (Peano's postulates) Let A be a set, and let $f\colon A \to A$ an injection which is not surjective (such sets are called *Dedekind-infinite*; c.f. problem 2). Fix $c \in A \setminus f[A]$.
   (a) Call $B \subset A$ *inductive* if $c \in B$ and $f[B] \subset B$. Let $N = \bigcap \{B \subset A \mid B \text{ is inductive}\}$. Show that N is inductive and that if $B \subset N$ is inductive then $B = N$.
   (b) Show that $f\colon N \to N$ is injective and has range exactly $N \setminus \{c\}$.
4. Let A, f, c be as in problem 3.
   (a) Show that there is a unique function $h\colon \omega \to A$ such that $h(0) = A$ and $h(n^+) = f(h(n))$.
   (b) Show that the image of h is N.
   (c*) Conclude that the Peano axioms have a unique model.
5. Let T be a transitive set. Show that $\bigcup T \subset T$.
6. What is wrong with the following argument:

   THEOREM 5. *Every set A is contained in a minimal transitive set.*

   PROOF. Recursively define a function f on ω as follows: $f(0) = A$ and $f(n^+) = \bigcup f(n)$. Finally let $S = \bigcup_{n=0}^{\infty} f(n)$. Then S is transitive: let $x \in y \in S$. Then $y \in f(n)$ for some n and then $x \in f(n^+) \subset S$. Also, if $A \subset T$ is a transitive we have that $f(0) \subset T$. Suppose that $f(n) \subset T$. Then $f(n^+) = \bigcup f(n) \subset \bigcup T \subset T$. It follows by inductiont that $f(n) \subset T$ for all T and hence that $S \subset T$. $\square$