Lior Silberman's Set Theory: Problem set 2 on the relations and functions

1. Show that $A \times \bigcup \mathcal{B} = \bigcup \{A \times B \mid B \in \mathcal{B}\}$ (in particular, you need to show that $\{A \times B \mid B \in \mathcal{B}\}$ is a set!)
2. A relation R is *transitive* if $\forall x, y, z : xRy \wedge yRz \to xRz$.
   (a) Show that the intersection of a set of transitive relations is a transitive relation.
   (b) Is the union of a set of transitive relations transitive?
3. (Transitive closure) Let R be a relation on a set A (recall that this means that $\operatorname{fld} R \subset A$).
   (a) Show that there is a smallest transitive relation containing R.
   (b) Let $R^{(1)} = R$ and define recursively $R^{(n+1)} = R \circ R^{(n)}$. Show that $\bigcup_{n=1}^{\infty} R^{(n)}$ is a transitive relation, in fact the transitive relation of part (a).
   RMK We'll later formalize the natural numbers and the notion of definitions by recursion and proof by induction. For now don't worry about those issues.
4. (Extension of functions) Let f, g be functions.
   (a) Show that $f = g$ iff $\operatorname{Dom} f = \operatorname{Dom} g$ and $\forall x \in \operatorname{Dom} f : f(x) = g(x)$.
   (b) Show that $f \subset g$ iff $\operatorname{Dom} f \subset \operatorname{Dom} g$ and $\forall x \in \operatorname{Dom} f : f(x) = g(x)$.
   (c) Let $\mathcal{F}$ be a *chain* of functions, that is a set of functions that that for all $f, g \in \mathcal{F}$ either $f \subset g$ or $g \subset f$. Show that $\bigcup \mathcal{F}$ is a function.
5. Show that for every set A, $A^{\emptyset} = \{\emptyset\}$. Show that for every nonempty set A, $\emptyset^A = \emptyset$.
6. For any set X construct a bijection $\mathcal{P}X \to \{0,1\}^X$ so which is a group isomorphism if we treat $\{0,1\}$ as the field with two elements.