

Lior Silberman's Math 412: Problem set 9, due 19/11/2019

- P1. Recall that a *projection* is a linear map E such that $E^2 = E$. For each n construct a projection $E_n: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of norm at least n (\mathbb{R}^n is equipped with the Euclidean norm unless specified otherwise). Prove for yourself that the norm of an *orthogonal* projection is 1.

Difference and Differential Equations

P2. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Let $\underline{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- (a) Find S invertible and D diagonal such that $A = S^{-1}DS$.
 – Prove for yourself the formula $A^k = S^{-1}D^kS$.
 (b) Find a formula for $\underline{v}_k = A^k \underline{v}_0$, and show that $\frac{\underline{v}_k}{\|\underline{v}_k\|}$ converges for any norm on \mathbb{R}^2 .

RMK You have found a formula for Fibonacci numbers (why?), and have shown that the real number $\frac{1}{2} \left(\frac{1+\sqrt{5}}{2} \right)^n$ is exponentially close to being an integer.

RMK This idea can solve any *difference equation* (see PS10). We now apply this to solving *differential* equations.

1. We will analyze the differential equation $u'' = -u$ with initial data $u(0) = u_0, u'(0) = u_1$.

- (a) Let $\underline{v}(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$. Show that u is a solution to the equation iff \underline{v} solves

$$\underline{v}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{v}(t).$$

- (b) Let $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find formulas for W^n and express $\exp(Wt) = \sum_{k=0}^{\infty} \frac{W^k t^k}{k!}$ as a matrix whose entries are standard power series.

- (c) Sum the series and show that $u(t) = u_0 \cos(t) + u_1 \sin(t)$.

- (d) Find a matrix S such that $W = S \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} S^{-1}$. Evaluate $\exp(Wt)$ again, this time using

$$\exp(Wt) = S \left(\exp \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} \right) S^{-1}.$$

2. Consider the differential equation $\frac{d}{dt} \underline{v} = B \underline{v}$ where B is as in PS7 problem 1.

- (a) Find matrices S, D so that D is in Jordan form, and such that $B = SDS^{-1}$.

- (b) Find $\exp(tD)$ as in 1(b) by computing a formula for D^n and summing the series.

- (c) Find the solution such that $\underline{v}(0) = (0 \ 1 \ 1 \ 0)^t$.

3. Let $A = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ with $z \in \mathbb{C}$.

- (a) Find (and prove) a simple formula for the entries of A^n .

- (b) Use your formula to decide the set of z for which $\sum_{n=0}^{\infty} A^n$ converge, and give a formula for the sum.

- (c) Show that the sum is $(\text{Id} - A)^{-1}$ when the series converges.

Extra credit

4. For any matrix A show that $\sum_{n=0}^{\infty} z^n A^n$ converges for $|z| < \frac{1}{\rho(A)}$.

Supplementary problems

- A. Consider the map $\text{Tr}: M_n(F) \rightarrow F$.
- (a) Show that this is a continuous map.
 - (b) Find the norm of this map when $M_n(F)$ is equipped with the $L^1 \rightarrow L^1$ operator norm (see PS8 Problem 2(a)).
 - (c) Find the norm of this map when $M_n(F)$ is equipped with the Hilbert–Schmidt norm (see PS8 Problem 4).
 - (*d) Find the norm of this map when $M_n(F)$ is equipped with the $L^p \rightarrow L^p$ operator norm. Find the matrices A with operator norm 1 and trace maximal in absolute value.
- B. Call $T \in \text{End}_F(V)$ *bounded below* if there is $K > 0$ such that $\|T\underline{v}\| \geq K \|\underline{v}\|$ for all $\underline{v} \in V$.
- (a) Let T be bounded below. Show that T is invertible, and that T^{-1} is a bounded operator.
 - (*b) Suppose that V is finite-dimensional. Show that every invertible map is bounded below.
- C. (The supremum norm and the Weierstrass M -test) Let V be a complete normed space.
- DEF For a set X call $f: X \rightarrow V$ *bounded* if there is $M > 0$ such that $\|f(x)\|_V \leq M$ for all $x \in X$ in which case we write $\|f\|_{\infty} = \sup_{x \in X} \|f(x)\|_V$ (equivalently, f is bounded if $x \mapsto \|f(x)\|_V$ is in $\ell^{\infty}(X)$).
- (a) Show that $\ell^{\infty}(X; V)$ is a vector space (this doesn't use completeness of V).
 - (b) Show that $\ell^{\infty}(X; V)$ is complete.
- DEF Now suppose that X is a topological space (if you aren't sure about this, simply assume $X \subset \mathbb{R}^n$). Let $C(X; V)$ denote the space of *continuous* functions $X \rightarrow V$ and let $C_b(X; V) = C(X; V) \cap \ell^{\infty}(X; V)$ be the space of *bounded* continuous functions, the latter equipped with the ℓ^{∞} -norm.
- (c) Show that $C_b(X; V)$ is complete (equivalently, that it is a closed subspace of $\ell^{\infty}(X; V)$).
- COR Deduce Weierstrass's M -test: $f_n: X \rightarrow V$ are continuous and $\|f_n\|_{\infty} \leq M_n$ with $\sum_n M_n < \infty$ then $\sum_n f_n$ converges to a continuous function bounded by $\sum_n M_n$.