Math 100 – SOLUTIONS TO WORKSHEET 22
L'HÔPITAL'S RULE

(1) Evaluate $\lim_{x\to 1}\frac{\log x}{x-1}$.

Solution: Since $\lim_{x\to 1}\log x = \log 1 = 0$ and $\lim_{x\to 1} x-1 = 1-1 = 0$ and since both the numerator and denominator are differentiable we apply l'Hôpital's rule and get:

$$\lim_{x\to 1}\frac{\log x}{x-1} = \lim_{x\to 1}\frac{1/x}{1} = \lim_{x\to 1}\frac{1}{x} = 1\,.$$

(2) (Final, 2014) Evaluate $\lim_{x\to 0}\frac{\cos x - e^{x^2}}{x^2}$.

Solution: We apply l'Hôpital's rule twice to get:

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x - e^{x^2}}{x^2} &= \lim_{x\to 0}\frac{-\sin x - 2xe^{x^2}}{2x}\\ &= \lim_{x\to 0}\frac{-\cos x - 2e^{x^2} + 4x^2e^{x^2}}{2}\\ &= \frac{-\cos 0 - 2e^0 + 0}{2} = -\frac{3}{2}\,.\end{aligned}$$

The use of the rule is justified since $\lim_{x\to 0}\left(\cos x - e^{x^2}\right) = \cos 0 - e^0 = 0$, $\lim_{x\to 0}\left(-\sin x - 2xe^{x^2}\right) = -\sin 0 - 0 = 0$ and $\lim_{x\to 0} x^2 = \lim_{x\to 0} 2x = 0$.

(3) Do (2) using a 2nd-order Taylor expansion.

Solution: To second order we have $\cos x \approx 1 - \frac{x^2}{2}$ and $e^u \approx 1 + u + \frac{u^2}{2}$ so $e^{x^2} \approx 1 + x^2$. We therefore have $\cos x - e^{x^2} \approx \left(1 - \frac{x^2}{2}\right) - \left(1 + x^2\right) = -\frac{3}{2}x^2$. We therefore have $\frac{\cos x - e^{x^2}}{x^2} \approx -\frac{3}{2}$ to zeroth order.

(4) (Final, 2015) Evaluate $\lim_{x\to 0}\frac{\log(1+x) - \sin x}{x^2}$.

Solution: We apply l'Hôpital's rule twice to get:

$$\lim_{x\to 0}\frac{\log(1+x) - \sin x}{x^2} = \lim_{x\to 0}\frac{\frac{1}{1+x} - \cos x}{2x} = \lim_{x\to 0}\frac{-\frac{1}{(1+x)^2} + \sin x}{2} = -\frac{1}{2}\,.$$

The use of the rule is justified since $\lim_{x\to 0}\left(\log(1+x) - \sin x\right) = \log 1 - \sin 0 = 0$, $\lim_{x\to 0}\left(\frac{1}{1+x} - \cos x\right) = \frac{1}{1} - \cos 0 = 0$ and $\lim_{x\to 0} x^2 = \lim_{x\to 0} 2x = 0$.

Remark: To third order we have $\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$ and $\sin x = x - \frac{x^3}{6}$ so, to first order,

$$\frac{\log(1+x) - \sin x}{x^2} \approx \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \left(x - \frac{x^3}{6}\right)}{x^2} = \frac{-\frac{x^2}{2} + \frac{1}{2}x^3}{x^2} = -\frac{1}{2} + \frac{1}{2}x \xrightarrow[x\to 0]{} \frac{1}{2}\,.$$

(5) Given that $f(2) = 5$, $g(2) = 3$, $f'(2) = 7$ and $g'(2) = 4$ find $\lim_{x\to 3}\frac{f(2x-4) - g(x-1) - 2}{g(x^2-7) - 3}$.

Solution: Since f, g are differentiable at 2, they are continuous there and

$$\begin{aligned}\lim_{x\to 3}\left(f(2x-4) - g(x-1) - 2\right) &= f(6-4) - g(3-1) - 2 = f(2) - g(2) - 2 = 5 - 3 - 2 = 0\\ \lim_{x\to 3}\left(g(x^2-7) - 3\right) &= g(9-7) - 3 = g(2) - 3 = 3 - 3 = 0\,.\end{aligned}$$

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By arithmetic of derivatives the numerator and denominator are differentiable at $x = 3$ and we may therefore apply l'Hopital's rule:

$$\lim_{x\to 3} \frac{f(2x-4) - g(x-1) - 2}{g(x^2-7) - 3} = \lim_{x\to 3} \frac{2f'(2x-4) - g'(x-1)}{2xg'(x^2-7)}$$
$$= \frac{2f'(2) - g'(2)}{2\cdot 3\cdot g'(2)} = \frac{2\cdot 7 - 4}{6\cdot 4} = \frac{10}{24} = \frac{5}{12}\,.$$

(6) Evaluate $\lim_{x\to 0^+} \frac{e^x}{x}$.

Solution: Since $e^x \xrightarrow[x\to 0]{} e^0 = 1$ while $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ we have $\lim_{x\to 0^+} \frac{e^x}{x} = \infty$.

(7) Evaluate $\lim_{x\to\infty} x^2 e^{-x}$.

Solution: We have $x^2 e^{-x} = \frac{x^2}{e^x}$ and as $x \to \infty$ both numerator and denominator diverge to ∞. The same holds for $2x$. We may therefore apply l'Hôpital's rule twice and get:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

(8) Evaluate $\lim_{x\to 0^+} x\log x$.

Solution: We have $x \log x = \frac{\log x}{1/x}$ and as $x \to 0^+$ both numerator and denominator diverge ($\log x$ to $-\infty$, $\frac{1}{x}$ to ∞). We may therefore apply l'Hôpital's rule twice and get:

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0\,.$$

(9) Evaluate $\lim_{x\to 0} (2x+1)^{1/\sin x}$.

Solution: We have $(2x+1)^{1/\sin x} = e^{\frac{\log(2x+1)}{\sin x}}$. Now since the function e^u is continuous it's enough to compute $\lim_{x\to 0} \frac{\log(2x+1)}{\sin x}$. As $x \to 0$, $\log(2x+1) \to \log 1 = 0$ and $\sin x \to \sin 0 = 0$ so we may apply l'Hôpital's rule and get

$$\lim_{x\to 0} \frac{\log(2x+1)}{\sin x} = \lim_{x\to 0} \frac{\frac{2}{2x+1}}{\cos x} = \lim_{x\to 0} \frac{2}{(2x+1)\cos x} = \frac{2}{\cos 0} = 2\,.$$

We therefore have

$$\lim_{x\to 0} (2x+1)^{1/\sin x} = \lim_{x\to 0} e^{\frac{\log(2x+1)}{\sin x}} e^{\lim_{x\to 0} \frac{\log(2x+1)}{\sin x}} = e^2\,.$$

(10) Evaluate $\lim_{x\to\infty} x^n e^{-x}$.

Solution: We note that for every $k > 0$, $\lim_{x\to\infty} x^k = \infty$ and similarly $\lim_{x\to\infty} e^x = \infty$. We therefore apply l'Hôpital's rule n times to get:

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0\,.$$

(11) Suppose $a > 0$. Evaluate $\lim_{x\to\infty} x^{-a} \log x$.

Solution: We have $x^{-a}\log x = \frac{\log x}{x^a}$ and as $x \to \infty$ both numerator and denominator diverge to ∞. We may therefore apply l'Hôpital's rule get:

$$\lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{x\to\infty} \frac{1/x}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{ax^a} = 0$$

since $a > 0$.