Math 100 – SOLUTIONS TO WORKSHEET 14
TAYLOR EXPANSION

1. The Linear Approximation

(1) Use a linear approximation to estimate
  (a) $\sqrt{1.2}$
  Solution: Let $f(x) = \sqrt{x}$ so that $f'(x) = \frac{1}{2\sqrt{x}}$. Then $f(1) = 1$ and $f'(1) = \frac{1}{2}$ so $f(1.2) \approx f(1) + f'(1) \cdot 0.2 = 1 + \frac{1}{2} \cdot 0.2 = 1.1$. Better: $f(1.21) = 1.1$ and $f'(1.21) = \frac{1}{2.2}$ so $f(1.2) = f(1.21 - 0.01) \approx 1.1 - 0.01 \cdot \frac{1}{2.2} \approx 1.09545$.
  (b) (Final, 2015) $\sqrt{8}$
  Solution: Using the same f we have $f(9-1) \approx f(9) + f'(9) \cdot (-1) = 3 - \frac{1}{6} = 2\frac{5}{6}$.
  (c) (Final, 2016) $(26)^{1/3}$
  Solution: Let $f(x) = x^{1/3}$ so that $f'(x) = \frac{1}{3}x^{-2/3}$. Then $f(27) = 3$ and $f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}$ so

$$f(26) = f(27-1) \approx f(27) + (-1) \cdot f'(27) = 3 - \frac{1}{27} = 2\frac{26}{27}.$$

  (d) $\log 1.07$
  Solution: Let $f(x) = \log x$ so that $f'(x) = \frac{1}{x}$. Then $f(1) = 0$ and $f'(1) = 1$ so $f'(1.1) \approx 0.07$.

2. Taylor approximation

(2) Let $f(x) = e^x$
  (a) Find $f(0), f'(0), f^{(2)}(0), \cdots$
  (b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.
  (c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.
  (d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.
  (e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.
  Solution: $f(x) = f'(x) = f^{(2)}(x) = \cdots = e^x$ so $f(0) = f'(0) = f''(0) = \cdots = 1$. Now $T_0(x) = 1$ works, as does $T_1(x) = 1 + x$. If $T_2(x) = 1 + x + cx^2$ then $T_2''(x) = 2c = 1$ means $c = \frac{1}{2}$ and $T_2(x) = 1 + x + \frac{1}{2}x^2$. Finally, $T_3(x) = 1 + x + \frac{1}{2}x^2 + dx^3$ works if $6d = 1$ so if $d = \frac{1}{6}$.

(3) Do the same with $f(x) = \ln x$ about $x = 1$.
  Solution: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ so $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$. Try $T_3(x) = a + bx + cx^2 + dx^3$ (can truncate later). Need $a = 0$ to make $T_3(x) = 0$. Diff we get $T_3'(x) = b + 2cx + 3dx^2$, setting $x = 0$ gives $b = 1$. Diff again gives $T_3''(x) = 2c + 6dx$ so $2c = -1$ and $c = -\frac{1}{2}$. Diff again give $T_3'''(x) = 6d = 2$ so $d = \frac{1}{3}$ and $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$. Truncate this to get T_0, T_1, T_2.

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)
  Solution: $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f^{(3)}(x) = \frac{6}{(1-x)^4}$, $f^{(4)}(x) = \frac{24}{(1-x)^5}$ $f^{(k)}(0) = k!$ and the Taylor expansion is $1 + x + x^2 + x^3 + x^4$.

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(5) Find the nth order expansion of $\cos x$.

Solution: $(\cos x)' = -\sin x$, $(\cos x)^{(2)} = -\cos x$, $(\cos x)^{(3)} = \sin x$, $(\cos x)^{(4)}(x) = \cos x$ and the pattern repeats. Plugging in zero we see that the derivatives at 0 (starting with the zeroeth) are $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ so the Taylor expansion is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

Solution: We have $c_2 = \frac{f^{(2)}}{2!} = 12$ so $f^{(2)} = 24$.

3. New from old

(7) (Final, 2016) Find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

Solution: Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f^{(2)}(x) = -\sin x$ and $f^{(3)}(x) = -\cos x$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$ and the third-order expansion of $\sin x$ is $0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 = x - \frac{1}{6}x^3$. We then have, correct to third order, that

$$(x+1)\sin x \approx (x+1)\left(x - \frac{1}{6}x^3\right) = x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 \approx x + x^2 - \frac{1}{6}x^3 .$$

(7) Find the 3rd order Taylor expansion of $\sqrt{x} + 3x$ about $x = 4$.

Solution: Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$, $f^{(2)}(x) = -\frac{1}{4x^{3/2}}$ and $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$. Thus $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f^{(2)}(4) = -\frac{1}{32}$, $f^{(3)}(4) = \frac{3}{256}$ and the third-order expansions are

$$\begin{aligned}\sqrt{x} &\approx 2 + \frac{1}{4}(x-4) - \frac{1}{32 \cdot 2!}(x-4)^3 + \frac{3}{256 \cdot 3!}(x-4)^3 \\ 3x &\approx 12 + 3(x-4)\end{aligned}$$

so that

$$\sqrt{x} + 3x \approx 14 + 3\frac{1}{4} \cdot (x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 .$$

(8) Find the 8th order expansion of $f(x) = e^{x^2} + \cos(2x)$. What is $f^{(6)}(0)$?

Solution: To fourth order we have $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24}$ so $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}$. We also know that $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720} + \frac{u^8}{40320}$ so $\cos(2x) = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8$ so

$$\begin{aligned}e^{x^2} + \cos(2x) &\approx \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}\right) + \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8\right) \\ &= 2 - x^2 + \frac{7}{6}x^4 + \frac{7}{90}x^6 + \frac{121}{2520}x^8 .\end{aligned}$$

In particular, $\frac{f^{(6)}(0)}{6!} = \frac{7}{90}$ so $f^{(6)}(0) = 6! \cdot \frac{7}{90} = \frac{720 \cdot 7}{90} = 56$.

(9) Show that $\log\frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x.

Solution: Let $f(x) = \log(1+x)$. Then $f'(x) = \frac{1}{1+x}$, $f^{(2)}(x) = -\frac{1}{(1+x)^2}$, $f^{(3)}(x) = \frac{1 \cdot 2}{(1+x)^3}$, $f^{(4)}(x) = -\frac{1 \cdot 2 \cdot 3}{(1+x)^4}$ and so on, so $f^{(k)}(x) = (-1)^{k-1} \cdot \frac{(k-1)!}{(1+x)^k}$. We thus have that $f(0) = 0$ and for $k \geq 1$ that $f^{(k)}(0) = (-1)^{k-1}(k-1)!$. Then $\frac{f^{(k)}(0)}{k!} = \frac{(-1)^{k-1}}{k}$ so

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

Plugging $-x$ we get:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

so

$$\log\frac{1+x}{1-x} = \log(1+x) - \log(1-x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots .$$

In particular

$$\log 3 = \log \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 2\left(\frac{1}{2} + \frac{1}{24} + \frac{1}{160} + \cdots\right) = 1 + \frac{1}{12} + \frac{1}{80} + \cdots \approx 1.096$$