Math 100 – SOLUTIONS TO WORKSHEET 13
RELATED RATES

(1) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $\left(1, \sqrt{3}\right)$ we have $\frac{dy}{dt} = 1$. Find $\frac{dx}{dt}$.

Solution: We differentiate along the curve with respect to time, finding

$$2y\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 2\frac{dx}{dt}\,.$$

Plugging in $\frac{dy}{dt} = 1$, $x = 1$, $y = \sqrt{3}$ we find: $2\sqrt{3} = 5\frac{dx}{dt}$ so at that time we have

$$\boxed{\frac{dx}{dt} = \frac{2\sqrt{3}}{5}\,.}$$

(2) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

Solution: The water fills a conical volume inside the drain. Suppose that at time t the height of the water is $h(t)$ and the radius at the surface of the water is $r(t)$. Then by similar triangles

$$\frac{r(t)}{h(t)} = \frac{1}{6}\,.$$

We therefore have $r(t) = \frac{h(t)}{6}$. The volume of the water is therefore

$$V(t) = \frac{1}{3}\pi r^2 h = \frac{\pi}{108}h^3(t)\,.$$

Differentiating we find

$$\frac{dV}{dt} = \frac{\pi}{36}h^2(t)\frac{dh}{dt}\,.$$

In particular, if $\frac{dV}{dt} = 5\text{m}^3/\text{min}$ and $h = 5\text{m}$ then

$$\frac{dh}{dt} = \frac{36\cdot 5}{\pi\cdot 5^2} = \frac{36}{5\pi}\frac{\text{m}}{\text{min}}\,.$$

(b) The drain is unclogged and water begins to clear at the rate of $\frac{\pi}{4}\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

Solution: We are now given $\frac{dV}{dt} = -\frac{\pi}{4}\frac{\text{m}^3}{\text{min}}$ and $\frac{dh}{dt} = -1\frac{\text{m}}{\text{min}}$. Then

$$h(t) = \sqrt{\frac{36\frac{dV}{dt}}{\pi\frac{dh}{dt}}} = \sqrt{\frac{-36\pi}{4\pi(-1)}} = \sqrt{9} = 3\text{m}\,.$$

(3) Two ships are travelling near an island. The first is located 20km due west of it, The second is located 15km due south of it and is moving due south at 7km/h. How fast is the distance between the ships changing if:

(a) The first ship is moving due north at 5km/h.

Solution: Suppose the ships are A, B at positions $(x_A, y_A), (x_B, y_B)$. Then the distance between them satisfies

$$D^2 = (x_A - x_B)^2 + (y_A - y_B)^2\,.$$

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It follows that

$$2D \cdot \frac{\mathrm{d}D}{\mathrm{d}t} = 2\left(x_A - x_B\right)\left(\frac{\mathrm{d}x_A}{\mathrm{d}t} - \frac{\mathrm{d}x_B}{\mathrm{d}t}\right) + 2\left(y_A - y_B\right)\left(\frac{\mathrm{d}y_A}{\mathrm{d}t} - \frac{\mathrm{d}y_B}{\mathrm{d}t}\right).$$

At the given time $\frac{\mathrm{d}x_A}{\mathrm{d}t} = \frac{\mathrm{d}x_B}{\mathrm{d}t} = 0$ (the ships are moving north/south), $y_A = 0$, $y_B = -15\text{km}$, $\frac{\mathrm{d}y_A}{\mathrm{d}t} = 5\text{km/h}$ and $\frac{\mathrm{d}y_B}{\mathrm{d}t} = -7\text{km/h}$. Finally at the given time $D = \sqrt{15^2 + 20^2}\text{km} = 25\text{km}$. At the given time we therefore have

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{(0 - (-15))}{25}\left(5 - (-7)\right) = \frac{15 \cdot 12}{25} = \boxed{7.2\text{km/h}}.$$

(b) The same setting, but now the first ship is moving toward the island.

Solution: Now $\frac{\mathrm{d}x_A}{\mathrm{d}t} = 5\text{km/h}$ but $\frac{\mathrm{d}x_B}{\mathrm{d}t} = 0$, $x_A = -20\text{km}$, $x_B = y_A = 0$, $y_B = -15\text{km}$, $\frac{\mathrm{d}y_A}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}y_B}{\mathrm{d}t} = -7\text{km/h}$. Again $D = 25\text{km}$. so

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{25}\left[(-20-0)(5-0) + (0-(-15))(0-(-7)\right] \\ &= \frac{1}{25}\left[-100+105\right] = \boxed{\frac{1}{5}\text{km/h}}.\end{aligned}$$