23. Antiderivatives (21/11/2019)

Goals.

Warning: Final Exam Locations to change.

(1) Idea of inverse operation
(2) Antiderivatives by massaging
(3) Antiderivatives of sums

Last Time.

l'Hôpital's rule: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (or $+\infty$, or $-\infty$)

and ~~$\lim_{x\to a} f'(x)$, $\lim_{x\to a} g'(x)$ exist~~ $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (or is $+\infty$ or $-\infty$)

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

(also works for $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$)

Warning: only applies if the limit is "indeterminate".

- can also be applied after some manipulation:

$x\log x = \frac{\log x}{1/x}$, $x^x = e^{x\log x}$.

Math 100 – WORKSHEET 23
ANTIDERIVATIVES

1. Warmup: inverse operations

(1) (Multiplication)
  (a) Calculate: $7 \times 8 = 56$
  (b) Find (some) a, b such that $ab = 15$.

take $a=1,\ b=15$
or $a=3,\ b=5$
or $\vdots$

(2) (Trig functions)
  (a) Calculate: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
  (b) Find all θ such that $\sin\theta = 1$.

$\arcsin 1 = \frac{\pi}{2}$, but $\sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$ for any $n \in \mathbb{Z}$

so the set of these θ is $\frac{\pi}{2} + 2\pi\mathbb{Z} = \left\{\frac{\pi}{2} + 2\pi n \mid n \in \mathbb{Z}\right\}$.

Date: 21/11/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Summary: (1) For each operation, computing the operation is (relatively) easy, ~~Finding~~ Computing reverse operasion is harder.

(2) reverse operation often multi-valued

(3) easy to check answer for the reverse operation.

Today: reverse differentiation

Problem: Given function g, find (some / all) f so that $f' = g$.

Worksheet 3

Fact: the ~~gene~~ general solution to $f' = g$ is $f + C$ where f is any particular solution, where C is an arbitrary constant.

(3) Simple differentiation

(a) Find one f such that $f'(x) = 1$.

$f(x) = x + 7$, $f(x) = x + 8$ ← "particular solutions"

"general solution"

(b) Find *all* such f.

$f(x) = x + C$, for some constant C.

(MVT says: if $h'(x) = 0$ for all x then h is constant)

(c) Find the f such that $f(7) = 3$.

Want $7 + C = 3$, so $C = -4$,

and $f(x) = x - 4$ works

2. Antidifferentiation by massaging

(4) Find f such that $f'(x) = 2x^3$.

Note: $(x^4)' = 4x^3$, $\underbrace{\text{dividing by } 2}_{\text{"~~massag~~" "massaging"}}$ we find

$$\frac{d}{dx}\left(\frac{1}{2}x^4\right) = 2x^3$$

so $\frac{1}{2}x^4$ works.

(5) Find f such that $f'(x) = -\frac{1}{x}$.

Note: $(\log|x|)' = \frac{1}{x}$ so $(-\log|x|)' = -\frac{1}{x}$

need $|\cdot|$ so the antiderivative has same domain as $\frac{1}{x}$.

(6) Find all f such that $f'(x) = \cos 3x$.

Note: $(\sin x)' = \cos x$, so $(\sin 3x)' = 3\cos 3x$

so $\left(\frac{1}{3}\sin(3x)\right)' = \cos 3x$ and the general solution is

$$\boxed{\frac{1}{3}\sin(3x) + C}$$

3. COMBINATIONS

(7) (Final, 2015) Find a function $f(x)$ such that $f'(x) = \sin x + \frac{2}{\sqrt{x}}$ and $f(\pi) = 0$.

$(-\cos x)' = \sin x$, Also, $(x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$. Thus $(-\cos x + 4x^{\frac{1}{2}})' = \sin x + \frac{2}{\sqrt{x}}$

The general solution is then $f(x) = -\cos x + 4\sqrt{x} + C$,

and $f(\pi) = -\cos\pi + 4\sqrt{\pi} + C = 1 + 4\sqrt{\pi} + C$.

So $f(\pi) = 0$ if $C = -1 - 4\sqrt{\pi}$, in which case

$$\boxed{f(x) = -\cos x + 4\sqrt{x} - 1 - 4\sqrt{\pi}}$$

(8) (Final, 2016) Find the general antiderivative of $f(x) = e^{2x+3}$.

Know: $\frac{d}{du}(e^u) = e^u$, so $\frac{d}{dx}(e^{2x+3}) = 2e^{2x+3}$ and the general antiderivative is $\boxed{\frac{1}{2}e^{2x+3} + C}$.

Alternative: write $e^{2x+3} = e^3 \cdot e^{2x}$, now $(e^x)' = e^x$, $(e^{2x})' = 2e^{2x}$,

so $\left(\frac{e^3}{2}e^{2x}\right)' = e^3 e^{2x}$.

(9) Find f such that $f'(x) = \frac{6x^4 - 2x - 2}{x^2}$.

We have $\dfrac{6x^4 - 2x - 2}{x^2} = 6x^2 - \dfrac{2}{x} - \dfrac{2}{x^2}$

Now $(x^3)' = 3x^2$, so $(2x^3)' = 6x^2$, $(\log|x|)' = \frac{1}{x}$, so $(-2\log|x|)' = -\frac{2}{x}$, and $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, so $\left(\frac{2}{x}\right)' = -\frac{2}{x^2}$. Putting things together

$$f(x) = 2x^3 - 2\log|x| + \frac{2}{x}$$

works.

(10) Find f such that $f'(x) = 2x^{1/3} - x^{-2/3}$ and $f(1000) = 5$.

Here, $(x^{4/3})' = \frac{4}{3}x^{1/3}$, so $\left(\frac{3}{2}x^{4/3}\right)' = 2x^{1/3}$ ← again, $\dfrac{d(u^n)}{du} = n u^{n-1}$

$(x^{1/3})' = \frac{1}{3}x^{-2/3}$ so $(-3x^{1/3})' = -x^{-2/3}$

so $f(x) = \frac{3}{2}x^{4/3} - 3x^{1/3} + C$

~~for~~ Want $5 = f(1000) = \frac{3}{2}(1000)^{4/3} - 3(1000)^{1/3} + C = 15{,}000 - 30 + C$

So $C = -14{,}965$ and

$$\boxed{f(x) = \frac{3}{2}x^{4/3} - 3x^{1/3} - 14{,}965}$$

(11) Find f such that $f''(x) = \sin x + \cos x$, $f(0) = 0$ and $f'(0) = 1$.

Note: $(-\cos x + \sin x + C)' = \sin x + \cos x$

so $f'(x) = -\cos x + \sin x + C$

also, $(-\sin x - \cos x + Cx + D)' = -\cos x + \sin x + C$

so $f(x) = -\sin x - \cos x + Cx + D$ for some C, D

$f'(0) = 1$ forces $1 = -1 + C$ so $C = 2$,

$f(0) = 0$ forces $0 = -1 + 2 \cdot 0 + D$ so $D = 1$

ans

$$\boxed{f(x) = -\sin x - \cos x + 2x + 1}$$