Math 100

L'Hopital's Rule

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x\to 1} x+1 = 2$$

How about $\lim_{x\to 0} \frac{\sin x}{x}$?

Theorem (L'Hopital's Rule)

Let f, g be differentiable near $x=a$.

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ $\underline{(\pm\infty)}$

and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$

Rmk : a can be $\pm\infty$ and L can be $\pm\infty$

Example : Find $\lim_{x\to 0} \frac{\sin x}{x}$

Let $f(x) = \sin x$, $g(x) = x$ and both diff. near 0.

$\boxed{\lim_{x\to 0} f(x) = 0 \qquad \lim_{x\to 0} g(x) = 0}$ ①

② $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$

So by L'Hopital's Rule, we have

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1$$

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty} .$$

Example $\lim_{x \to \infty} \frac{x}{e^x}$

Since $\lim_{x \to \infty} x = \infty$, $\lim_{x \to \infty} e^x = \infty$ and both diff.

by L'Hopital's Rule we have

$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0 .$$

Math 100 – WORKSHEET 22
L'HÔPITAL'S RULE

(1) Evaluate $\lim_{x\to 1}\frac{\log x}{x-1}$.

Since $\lim_{x\to 1}\log x = 0$ and $\lim_{x\to 1} x-1 = 0$ and both diff. near $x=1$, we apply L'Hopital's Rule to get

$$\lim_{x\to 1}\frac{\log x}{x-1} = \lim_{x\to 1}\frac{\frac{1}{x}}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

(2) (Final, 2014) Evaluate $\lim_{x\to 0}\frac{\cos x - e^{x^2}}{x^2}$.

$\frac{\cos 0 - e^0 = 0}{0^2} = \frac{0}{0}$

Since $\lim_{x\to 0}\cos x - e^{x^2} = \cos 0 - e^0 = 1$ and $\lim_{x\to 0} x^2 = 0$, and both diff. near $x=0$, by L'Hopital's Rule we have

$$\lim_{x\to 0}\frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0}\frac{-\sin x - e^{x^2}\cdot 2x}{2x} \quad \left(\begin{array}{l}\lim_{x\to 0}(-\sin x - e^{x^2}\cdot 2x) = 0 \\ \lim_{x\to 0} 2x = 0\end{array}\right)$$

L'Hopital's Rule

product rule

$$\lim_{x\to 0}\frac{-\cos x - (e^{x^2}\cdot 2 + (e^{x^2}\cdot 2x)\cdot 2x)}{2} = \frac{-\cos 0 - (e^0\cdot 2 + 0)}{2} = \frac{-1-2}{2} = \frac{-3}{2} \blacksquare$$

(3) Do (2) using a 2nd-order Taylor expansion.

Since $\cos x = 1 - \frac{x^2}{2!} + \cancel{\frac{x^4}{4!}} - \cancel{\frac{x^6}{6!}} + \dots$

$e^{x^2} = 1 + x^2 + \cancel{\frac{(x^2)^2}{2!}} + \cancel{\frac{(x^2)^3}{3!}} + \dots \quad \left(e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots\right)$

we have $\cos x \approx 1 - \frac{x^2}{2}$ and $e^{x^2} \approx 1 + x^2 \Rightarrow \cos x - e^{x^2} \approx 1 - \frac{x^2}{2} - (1+x^2) = -\frac{3}{2}x^2$

therefore $\lim_{x\to 0}\frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0}\frac{-\frac{3}{2}x^2}{x^2} = -\frac{3}{2}$

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(4) (Final, 2015) Evaluate $\lim_{x\to 0} \frac{\log(1+x)-\sin x}{x^2}$.

Since $\lim_{x\to 0} \log(1+x) - \sin x = \log 1 - \sin 0 = 0$ and $\lim_{x\to 0} x^2 = 0$

are both diff. near $x=0$, we apply L'Hopital's Rule to get

$$\lim_{x\to 0} \frac{\log(1+x)-\sin x}{x^2} \underset{\text{L'Hopital's Rule}}{=} \lim_{x\to 0} \frac{\frac{1}{1+x} - \cos x}{2x} \quad \left(\lim_{x\to 0} \frac{1}{1+x} - \cos x = \frac{1}{1+0} - \cos 0 = 0, \ \lim_{x\to 0} 2x = 0 \right)$$

$$\lim_{x\to 0} \frac{-\frac{1}{(1+x)^2} - (-\sin x)}{2} = \frac{-\frac{1}{(1+0)^2} - (-\sin 0)}{2} = \frac{-1}{2}$$

(5) Given that $f(2) = 5$, $g(2) = 3$, $f'(2) = 7$ and $g'(2) = 4$ find $\lim_{x\to 3} \frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3}$.

Since $\lim_{x\to 3} \boxed{f(2x-4)} - g(x-1) - 2 = f(2) - g(2) - 2 = 5-3-2 = 0$

$\lim_{x\to 3} g(x^2-7) - 3 = g(9-7) - 3 = g(2) - 3 = 3-3 = 0$

and both diff. near $x=3$, we can apply L'Hopital's Rule to get

$$\lim_{x\to 3} \frac{f(2x-4) - g(x-1) - 2}{g(x^2-7)-3} = \lim_{x\to 3} \frac{\overset{\text{chain rule}}{f'(2x-4)\cdot 2} - g'(x-1)\cdot 1}{g'(x^2-7)\cdot 2x}$$

$$= \frac{f'(2)\cdot 2 - g'(2)}{g'(2)\cdot 2\cdot 3} = \frac{7\cdot 2 - 4}{4\cdot 2\cdot 3} = \frac{10}{24}$$

chain rule

(6) Evaluate $\lim_{x\to 0^+} \frac{e^x}{x}$.

$\lim_{x\to 0^+} e^x = 1$

$\left.\lim_{x\to 0^+} x = 0 \text{ and } \lim_{x\to 0^+} \frac{1}{x} = \infty \right\} \Rightarrow \lim_{x\to 0^+} \underset{1}{e^x} \cdot \underset{\infty}{\frac{1}{x}} = \infty$

$\boxed{\lim_{x\to 0^+} \frac{e^x}{x} = \lim_{x\to 0^+} \frac{e^x}{1} = 1 \ \times \ \text{Wrong. Can't apply}}$

L'Hopital's Rule

So far we only work with case $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

When dealing with cases like

$$0 \cdot (\pm\infty) \;/\; 1^{\infty} \;/\; 0^{0} \;/\; \infty^{0}$$

$$\boxed{\text{Methods: Always Convert into } \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ and then apply L'Hopital's Rule}}$$

Example

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

$$\boxed{x = e^{\log x}, \; \cancel{x>0} \text{ for all } x>0}$$

∞^{0}

$\downarrow$

$\frac{\infty}{\infty}$

$$x^{\frac{1}{x}} = \left(e^{\log x}\right)^{\frac{1}{x}} = e^{\frac{\log x}{x}}$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\log x}{x}} = e^{\boxed{\lim_{x\to\infty}\frac{\log x}{x}}} = e^{0} = 1$$

$$\lim_{x\to\infty}\frac{\log x}{x} \overset{\text{L'Hopital's Rule}}{=\!=\!=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

$$\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$$

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$\infty \cdot 0$

(7) Evaluate $\lim_{x\to\infty} x^2 e^{-x}$.

$\left(e^{-x} = \frac{1}{e^x} \right)$

Note $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$

Since $\lim_{x\to\infty} x^2 = \lim_{x\to\infty} e^x = \infty$ and both diff.

by L'Hopital's Rule $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$ $\left(\lim_{x\to\infty} 2x = \lim_{x\to\infty} e^x = \infty \right)$

$\overset{\text{L'Hopital's}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$.

(8) Evaluate $\lim_{x\to 0^+} x \log x$.

$0 \quad \log 0^+ = -\infty$

$\left(x = \frac{1}{1/x} \right)$

Note $\lim_{x\to 0^+} x \log x = \lim_{x\to 0^+} \frac{\log x}{1/x}$ $\left(\lim_{x\to 0} \frac{x}{1/\log x} \right)$

Since $\lim_{x\to 0^+} \log x = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and both diff.

by L'Hopital's Rule, we have $\lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{x^2}} \cdot \frac{x^2}{x^2}$

$= \lim_{x\to 0^+} -x = 0$

(9) Evaluate $\lim_{x\to 0} (2x+1)^{1/\sin x}$.

Notice $2x+1 = e^{\log(2x+1)}$

then $\lim_{x\to 0} (2x+1)^{\frac{1}{\sin x}} = \lim_{x\to 0^+} \left(e^{\log(2x+1)} \right)^{\frac{1}{\sin x}}$

$= \lim_{x\to 0^+} e^{\frac{\log(2x+1)}{\sin x}} = e^{\lim_{x\to 0^+} \frac{\log(2x+1)}{\sin x}} = e^2$

Since $\lim_{x\to 0^+} \log(2x+1) = 0$ and $\lim_{x\to 0^+} \sin x = 0$ and both diff.,

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chain rule

by L'Hopital's Rule, $\lim_{x\to 0^+} \frac{\log(2x+1)}{\sin x} = \lim_{x\to 0^+} \frac{\frac{1}{2x+1} \cdot 2}{\cos x}$

$= \frac{\frac{1}{1} \cdot 2}{\cos 0} = 2$

n fixed ($n > 100$)

(10) Evaluate $\lim_{x \to \infty} x^n e^{-x}$.

Note $\quad X^n e^{-X} = \dfrac{X^n}{e^X}$

Since $\quad \lim_{X \to \infty} X^n = \lim_{X \to \infty} e^X = \infty$ and both diff.,

we apply L'Hôpital's Rule to get

$$\lim_{X \to \infty} \frac{X^n}{e^X} = \lim_{X \to \infty} \frac{n \cdot X^{n-1}}{e^X} = \lim_{X \to \infty} \frac{n(n-1)\, X^{n-2}}{e^X}$$

n-th ~~def~~ derivative

$$= \cdots = \lim_{X \to \infty} \frac{n(n-1)\cdots 1}{e^X} = 0 \qquad \left(\lim_{X \to \infty} X^k = \infty \text{ for } 1 \le k \le n \right).$$

(11) Suppose $a > 0$. Evaluate $\lim_{x \to \infty} x^{-a} \log x$.