19. Curve Sketching (7/11/2019)

Goals.

(1) Curve sketching protocol
(2) Examples from past exams

Last Time.

MVT $\Rightarrow$ shape of the graph of f. $\Rightarrow$ curve sketching

Last time: $f(x) = \frac{x^2-9}{x^2+3}$

$$\lim_{x\to\infty} \frac{x^2-9}{x^2+3} = \lim_{x\to\infty} \frac{1-9/x^2}{1+3/x^2} = 1,$$ same as $x\to-\infty$

[16] **4.** Let $f(x) = x\sqrt{3-x}$.

(a) (2 marks) Find the domain of $f(x)$.

Answer

$(-\infty, 3]$ or $\{x : x \leq 3\}$

Expression makes sense if $3-x \geq 0$ under the square root, i.e. if $x \leq 3$.

(b) (4 marks) Determine the x-coordinates of the local maxima and minima (if any) and intervals where $f(x)$ is increasing or decreasing.

$$f'(x) = \sqrt{3-x} + \frac{x}{2\sqrt{3-x}}(-1) = \frac{2(3-x)-x}{2\sqrt{3-x}} = \frac{6-3x}{2\sqrt{3-x}} = \frac{3}{2}\cdot\frac{2-x}{\sqrt{3-x}}.$$

f' is positive when $x<2$, so f is increasing there

negative when $2<x<3$, so f is decreasing there

$f'(2)=0$.

Because f' exists on all of $(-\infty,3)$, any local extremum would occur at a critical point. Since $f'>0$ if $x<2$, $f'<0$ if $x>2$ we see the only local extremum is a local maximum at $x=2$.

(c) (2 marks) Determine intervals where $f(x)$ is concave upwards or downwards, and the x-coordinates of inflection points (if any). You may use, without verifying it, the formula $f''(x) = (3x-12)(3-x)^{-3/2}/4$.

$$= \frac{3}{4}\,\frac{x-4}{(3-x)^{3/2}}$$

Observe if $x<3$, $\frac{3}{4}\frac{x-4}{(3-x)^{3/2}} < 0$, so f is concave down on its entire domain.

(possible error: notice that numerator vanishes at $x=4$, positive thereafter, forget about domain of f)

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Question 4 continued

(d) (2 marks) There is a point at which the tangent line to the curve $y = f(x)$ is vertical. Find this point.

$$\lim_{x\to 3^-} f'(x) = \lim_{x\to 3^-} \frac{3}{2}\cdot\frac{\cancel{2-x}}{\sqrt{3-x}} = -\infty$$

Answer $(3, f(3)) = (3,0)$

need a point in the plane

i.e. the line tangent to the graph at $(3,0)$ is vertical

(e) (2 marks) The graph of $y = f(x)$ has no asymptotes. However, there is a real number a for which $\lim_{x\to-\infty} \frac{f(x)}{|x|^a} = -1$. Find the value of a.

[thoughts: if x is huge, $3-x \approx |x|$ (3 is negligible) and $f \approx x\sqrt{|x|} \approx -|x|^{3/2}$.]

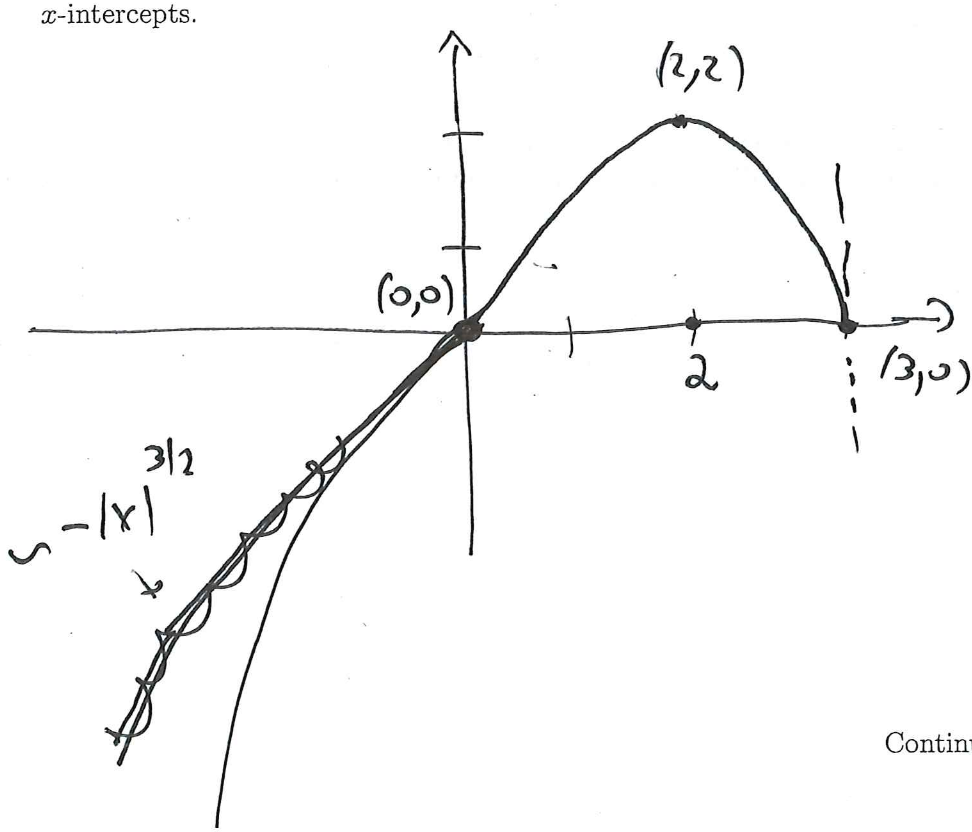
Answer $a = 3/2$

$$\lim_{x\to-\infty} \frac{f(x)}{|x|^{3/2}} = \lim_{x\to\infty} \frac{x\sqrt{3-x}}{|x|^{3/2}} = \lim_{x\to\infty} \frac{x}{|x|}\cdot\frac{\sqrt{3-x}}{\sqrt{|x|}} = \lim_{x\to-\infty} (-1)\sqrt{\frac{3}{|x|} - \frac{x}{|x|}} =$$

$$= -\lim_{x\to-\infty} \sqrt{1+\frac{3}{|x|}} = -\sqrt{1+0} = -1$$

(f) (4 marks) Sketch the graph of $y = f(x)$, showing the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above and also all x-intercepts.

$f(0) = 0$
$f(2) = 2$
$f(3) = 0$



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[14] 4. Let

$$f(x) = \begin{cases} \frac{4}{\pi}\tan^{-1} x, & \text{if } x \geq 1, \\ 2 - x^4, & \text{if } x < 1. \end{cases}$$

[*Note:* Another notation for $\tan^{-1}$ is arctan.]

(a) (3 marks) Show that $f(x)$ is continuous at $x = 1$.

$f(1) = \frac{4}{\pi}\arctan(1) = \frac{4}{\pi}\cdot\frac{\pi}{4} = 1.$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(2-x^4) = 2-1^4 = 1,$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}\left(\frac{4}{\pi}\arctan(x)\right) = \frac{4}{\pi}\arctan(1) = 1$

all equal $\Rightarrow f$ is continuous at $x=1$

(b) (1 mark) Determine the equations of any asymptotes (horizontal, vertical or slant).

[$\lim_{\theta\to\frac{\pi}{2}^-} \tan(\theta) = \infty$, so $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$]

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4}{\pi}\arctan(x) = \frac{4}{\pi}\cdot\frac{\pi}{2} = 2$, so have horizontal asymptote $y=2$ as $x\to\infty$.

No asymptote as $x\to-\infty$, or vertical ones.

(c) (4 marks) Determine all critical numbers, open intervals where f is increasing or decreasing, and the x-coordinates of all local maxima or local minima (if any).

$$f'(x) = \begin{cases} \frac{4}{\pi(1+x^2)} & x>1 \\ -4x^3 & x<1 \end{cases}$$

$f'(1)$ DNE: $\lim_{h\to0^-} \frac{f(1+h)-f(1)}{h} = \frac{d}{dx}\Big|_{x=1}(2-x^4) = -4$

$\lim_{h\to0^+} \frac{f(1+h)-f(1)}{h} = \frac{d}{dx}\Big|_{x=1}\frac{4}{\pi}\arctan(x) = \frac{2}{\pi\cdot2} = \frac{2}{\pi} \neq -4.$

f cts at 1

So $f'(x)>0$ if $x<0$ or $x>1$

$f'(x)<0$ if $0<x<1$

$f'(x)=0$ if $x=0$ (critical pt)

f' undef if $x=1$ (singular pt)

So f increasing on $(-\infty,0)$, $(1,\infty)$

decreasing on $(0,1)$

has local max at $x=0$

has local min at $x=1$

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Question 4 continued

(d) (2 marks) Determine open intervals where the graph of f is concave upwards or concave downwards, and the x-coordinates of all inflection points (if any).

$$f''(x) = \begin{cases} -\frac{4}{\pi} \cdot \frac{2x}{(1+x^2)^2} & x > 1 \\ -12x^2 & x < 1 \end{cases}$$

see $f''(x) < 0$ if $x \neq 0, 1$

so f is concave down on $(-\infty, 1)$, again on $(1, \infty)$

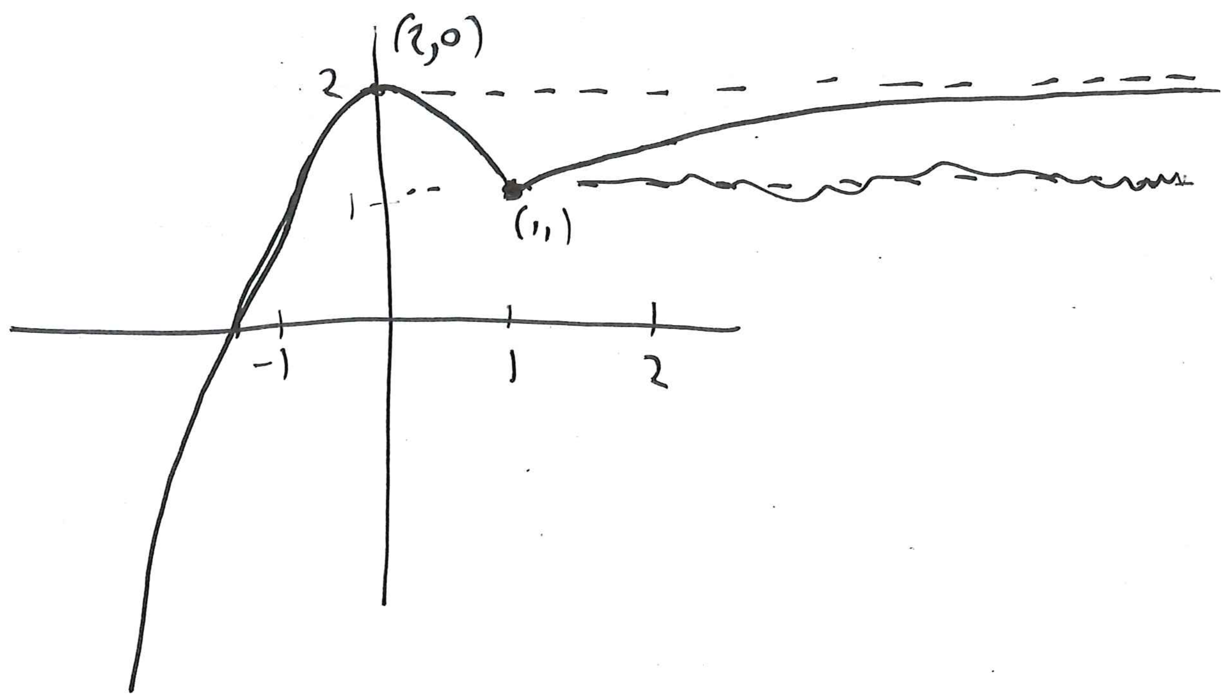
no inflection pts (no change in concavity)

(e) (4 marks) Sketch the curve $y = f(x)$, showing all the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above (if any).

$f(0) = 2$

$f(1) = 1$

$f(-2^{1/4}) = 0$



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