16. Minima and Maxima (29/10/2019)

Goals.

(1) Global and local extrema
(2) Critical and singular points
(3) Finding minima and maxima using differentiation

Last Time.

Taylor remainder estimates: if we expand f about $x=a$ then

$$R_n(x) = f(x) - T_n(x) = f_n(x) - \left(f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n\right)$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some (unknown) c between a and x.

(to make error estimates we bound $f^{(n+1)}(c)$ from above and below)

Optimization: Have some "objective function" (goal) depending on a parameter. Want value of parameter makes objective "best".

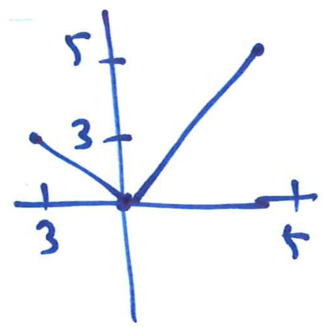
two parts: (1) construct objective function
(2) calculus: use derivatives to analyze this function.

Today: step 2.

1. Absolute minima and maxima by hand

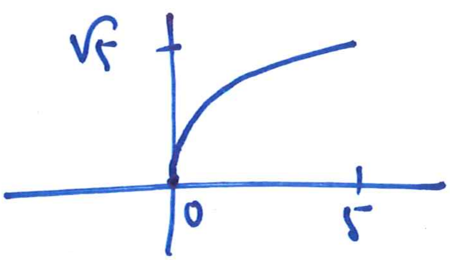
Theorem. *If f is continuous on $[a, b]$ it has an absolute maximum and minimum there.*

(1) Find the absolute maximum and minimum values of $f(x) = |x|$ on the interval $[-3, 5]$.



max is 5, attained at $x=5$
min is 0, attained at $x=0$

(2) Find the absolute maximum and minimum of $f(x) = \sqrt{x}$ on $[0, 5]$.

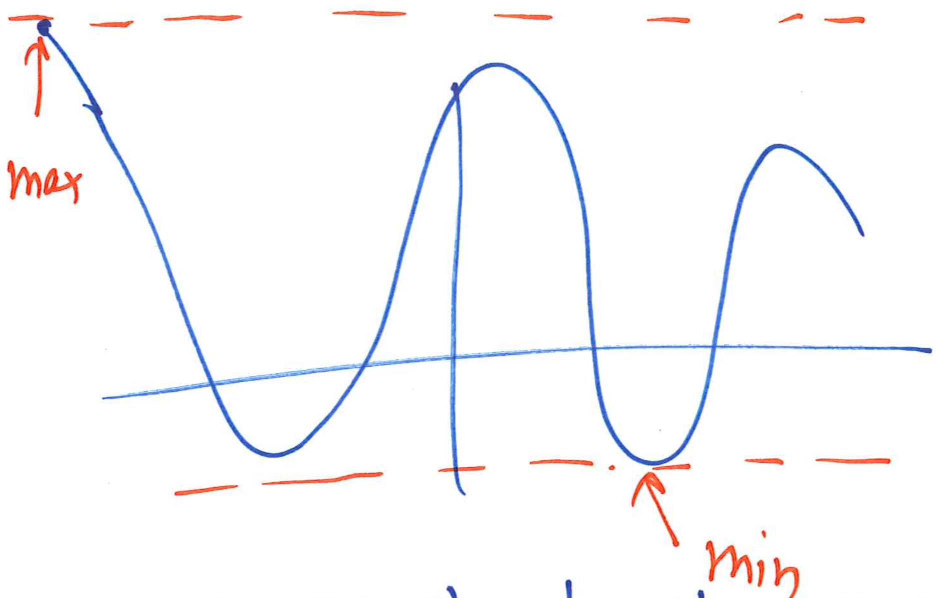


max is $\sqrt{5}$, attained at $x=5$
min is 0, attained at $x=0$

Date: 29/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Minima & Maxima

Definition: Let f be defined on $[a,b]$, let $c \in [a,b]$.
We say $f(c)$ is the (global) maximum of f on $[a,b]$
if $f(c) \geq f(x)$ for all x in $[a,b]$,
(global) minimum if $f(c) \leq f(x)$ for all such x



extremum = min or max

We call $f(c)$ then also the maximum/minimum <u>value</u> of f.
We say this "attained" at c.

Can use calculus to find extrema.

Def: We say c is a (point of) *local* ~~min~~ maximum of f if f is defined on both sides of c, and $f(x) \le f(c)$ for x *close* to c
(similarly for global min)

Thm: (Fermat) Calculus can detect local extrema: If f has a local extremum at c, *and* $f'(c)$ exists then $f'(c) = 0$

Pf: To the left of local max, secant lines have slope ≥ 0
To " right " " ", " " " " ≤ 0
so the limit of the slopes is both ≥ 0, ≤ 0 i.e. $= 0$

Conclusion: Suppose f is defined on $[a,b]$
local extrema occur ~~on~~ only at points c where:

(1) $f'(c) = 0$ (call c "critical point")

(2) $f'(c)$ undef (call c "singular points")

If f is *cts* on $[a,b]$ then it has global max & global min, these must occur in one of:

(1) at a critical point
(2) at a singular point
(3) at an endpoint of interval

2. Derivatives and local extrema

Theorem (Fermat). *If, in addition, f is defined and differentiable near c (on both sides!) and has a local extremum at c then $f'(c) = 0$.*

Procedure

- Call c a *critical point* if $f'(c) = 0$, a *singular point* if $f'(c)$ does not exist.
- To find absolute maximum/minimum of a continuous function f defined on $[a, b]$:
  - Evaluate $f(c)$ at all critical and singular point.
  - Evaluate $f(a), f(b)$.
  - Choose largest, smallest value.

(3) (Final, 2011) Let $f(x) = 6x^{1/5} + x^{6/5}$.

(a) Find the critical numbers and singularities of f.

$f'(x) = \frac{6}{5} x^{-4/5} + \frac{6}{5} x^{1/5}$ so $x=0$ is a singular point.

for $x \neq 0$, $f'(x) = \frac{6}{5} \cdot x^{-4/5}(1+x)$, and $x=-1$ is the only critical point.

(b) Find its absolute maximum and minimum on the internal $[-32, 32]$.

$f(-32) = 6 \cdot (-2) + (-2)^6 = 52$

$f(-1) = 6 \cdot (-1) + (-1)^6 = -5$

$f(0) = 0$

$f(32) = 6 \cdot 2 + 2^6 = 76$

By Fermat's theorem, the global maximum is 76

" minimum is -5

(4) (Final, 2015) Find the critical points of $f(x) = e^{x^3 - 9x^2 + 15x - 1}$

$f'(x) = f(x) \cdot (3x^2 - 18x + 15) = 3 f(x) (x^2 - 6x + 5) =$

$= 3 f(x) (x-1)(x-5)$

So $x = 1$, $x = 5$ are critical points. no others since $e^y \neq 0$ for all y.

(5) (caution)

(a) Show that $f(x) = (x-1)^4 + 7$ attains its absolute minimum at $x = 1$.

$f(1) = 7$, and for any x, $(x-1)^4 \geq 0$ (it's a square) so $f(x) \geq 7$.

(b) Show that $f(x) = (x-1)^3 + 7$ has $f'(1) = 0$ but has no local minimum or maximum there.

$f'(x) = 3(x-1)^2$ so $f'(1) = 0$ But: if $x > 1$, $(x-1)^3 > 0$ so $f(x) > 7$
if $x < 1$, $(x-1)^3 < 0$ so $f(x) < 7$.

so f has neither a local min nor a local max at $x=1$.

(6) (Midterm, 2010) Find the maximum value of $x\sqrt{1 - \frac{3}{4}x^2}$ on the interval $[0, 1]$.

Let $f(x) = x\sqrt{1-\frac{3}{4}x^2}$, [(0) f is cts on $[0,1]$. (def by formula)]

(1) $f'(x) = \sqrt{1-\frac{3}{4}x^2} + x\frac{1}{2\sqrt{1-\frac{3}{4}x^2}}\left(-\frac{3}{2}x\right)$

$$= \frac{2(1-\frac{3}{4}x^2) - \frac{3}{2}x^2}{2\sqrt{1-\frac{3}{4}x^2}} = \frac{2-3x^2}{2\sqrt{1-\frac{3}{4}x^2}}.$$

(2) for $0 \leq x \leq 1$, $\frac{3}{4}x^2 \leq \frac{3}{4}$, so $1-\frac{3}{4}x^2 > 0$, i.e. f has no singular points

f has critical points where $2-3x^2=0$, i.e. where $x^2 = 2/3$, i.e. where $x = \sqrt{\frac{2}{3}}$.

(3) $f(0) = 0$, $f\left(\sqrt{\frac{2}{3}}\right) = \sqrt{\frac{2}{3}}\cdot\sqrt{1-\frac{3}{4}\frac{2}{3}} = \sqrt{\frac{2}{3}}\cdot\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{3}}$. $f(1) = \sqrt{1-\frac{3}{4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

the largest of the three values is $\frac{1}{\sqrt{3}}$, so that is the maximum value of f on the interval.

Alternative solution:

Since $f(x) = x\sqrt{1-\frac{3}{4}x^2}$ is non negative on $[0,1]$,
it's enough to find maximum of $g(x) = f^2(x) = x^2\left(1-\frac{3}{4}x^2\right)$
then $g'(x) = 2x\left(1-\frac{3}{4}x^2\right) - 2x^2\left(\frac{3}{2}x\right) = x\left(2-3x^2\right)$. . .

(7) (Final, 2007) Let $f(x) = x\sqrt{3-x}$.

(a) Find the domain of f.

f is defined for $x \leq 3$. (on $(-\infty, 3]$.

(b) Determine the x-coordinates of any local maxima or minima of f.

$$f'(x) = \sqrt{3-x} + x \cdot \frac{1}{2\sqrt{3-x}} \cdot (-1) = \frac{2(3-x)-x}{2\sqrt{3-x}} = \frac{6-3x}{2\sqrt{3-x}}$$

so f has a critical point at $x=2$,

a singular point at $x=3$