# 15. Taylor Remainder (24/10/2019)

Goals.

(1) Review Taylor expansion
(2) Lagrange remainder for linear approximation
(3) Lagrange remainder: general case

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Last Time.

Linear & polynomial approximation: extrapolating information from $f(a), f'(a), f^{(2)}(a), \ldots$ to estimate $f(x)$ for $x$ near $a$.

Formula: $T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

memorize!

Question: why is $0! = 1$? Answer (1) $(n+1)! = n! \cdot (n+1)$ so $1! = 0! \cdot 1$

Answer 2: $n!$ = number of ways to order $n$ things
exactly one way to order 0 things

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Ex: In special relativity, the energy of a moving particle is $E(v) = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$ where $v$ is the velocity. What happens for small $v$?

consider $f(x) = (1-x)^{-\frac{1}{2}}$, so that $f'(x) = -\frac{1}{2}(1-x)^{-3/2}(-1) = \frac{1}{2}(1-x)^{-3/2}$ $\quad \frac{d(1-x)}{dx}$

same way $f''(x) = \frac{3}{4}(1-x)^{-5/2}$. So $f(0)=1$, $f'(0) = \frac{1}{2}$, $f''(0) = \frac{3}{4}$, so $f(x) \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2$

So $E(v) \approx mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2\right) = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2}$

Math 100 – WORKSHEET 15
TAYLOR REMAINDER ESTIMATES

# 1. Review: Taylor expansion

(1) Estimate $(4.1)^{3/2}$ using a linear and a quadratic approximation.

$\frac{3}{8} \cdot \frac{1}{2!}$

Let $f(x) = x^{3/2}$, then $f^{(1)}(x) = \frac{3}{2}x^{1/2}$, $f''(x) = \frac{3}{4}x^{-\frac{1}{2}}$

so $f(4) = 8$, $f^{(1)}(4) = 3$, $f^{(2)}(4) = \frac{3}{8}$, so $T_1(x) = 8 + 3(x-4)$

$T_2(x) = 8 + 3(x-4) + \frac{3}{16}(x-4)^2$

$T_1(4.1) = 8 + 3(4.1-4) = 8.3$

2015 Quiz $T_2(4.1) = 8.3 + \frac{3}{1600}$

(2) The third-order expansion of $h(x)$ about $x = 2$ is $3 + \frac{1}{2}(x-2) + 2(x-2)^3$. What are $h'(2)$ and $h''(2)$?

$h'(2) = \frac{1}{2}$, $h^{(2)}(0) = 0$ (no quadratic term)

(3) (Final, 2016) Find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

Method 1: let $f(x) = (x+1)\sin x$, diff 3 times

Method 2: let $g(x) = \sin x$, then $g'(x) = \cos x$, $g''(x) = -\sin x$, $g^{(3)}(x) = -\cos x$

so $g(0) = g^{(2)}(0) = 0$, $g^{(1)}(0) = 1$, $g^{(3)}(0) = -1$, so to third order

$\sin x \approx x - \frac{1}{6}x^3$

$\frac{1}{3!}$

so, to third order, $(x+1)\sin x \approx (x+1)(x - \frac{x^3}{6})$

$= x + x^2 - \frac{x^3}{6} - \cancel{\frac{x^4}{6}}$

## 2. Error estimate 1

Let $R_1(x) = f(x) - T_1(x)$ be the *remainder*. Then there is $c$ between $a$ and $x$ such that

$$R_1(x) = \frac{f^{(2)}(c)}{2!}(x-a)^2$$

(4) Estimate the error in the linear approximations to $(4.1)^{3/2}$.

$f^{(2)}(c) = \frac{3}{4}c^{-\frac{1}{2}}$. By Lagrange form of the remainder,

$$R_1(4.1) = \frac{1}{2!}\cdot\frac{3}{4}c^{-\frac{1}{2}}(4.1-4)^2 = \frac{3}{800}\cdot c^{-\frac{1}{2}} \text{ for some } c \text{ between } 4, 4.1$$

Since $c \geq 4$, $c^{-\frac{1}{2}} \leq 4^{-\frac{1}{2}} = \frac{1}{2}$ so $R_1(4.1) \leq \frac{3}{800}\cdot\frac{1}{2} = \frac{3}{1600}$

(also true: $c \geq 1$, so $c^{-\frac{1}{2}} \leq 1$, so error $\leq \frac{3}{800}$)

Also, $\frac{3}{800}c^{-\frac{1}{2}} > 0$ so $R_1(4.1) > 0$, i.e. $f(4.1) > T_1(4.1)$

(5) (Final, 2012) Show $-\frac{5}{32} \leq \log\left(\frac{8}{9}\right) \leq -\frac{1}{9}$ using the linear approximation to $f(x) = \log\left(1-x^2\right)$.

Note: $\frac{8}{9} = 1 - \frac{1}{9} = 1 - \left(\frac{1}{3}\right)^2$ so looking at $f\left(\frac{1}{3}\right)$.

$$f'(x) = \frac{-2x}{1-x^2}, \quad f''(x) = -\frac{2}{1-x^2} - \frac{4x^2}{(1-x^2)^2} = -\frac{2+2x^2}{(1-x^2)^2} = -\frac{2(1+x^2)}{(1-x^2)^2}$$

so $T_1(x) = f(0) + f'(0)x = 0 + 0x = 0$

and $R_1(x) = f(x) - T_1(x) = f(x)$, so $\log\left(\frac{8}{9}\right) = R_1\left(\frac{1}{3}\right)$

By the Lagrange form of the remainder, there is $0 < c < \frac{1}{3}$ so that

$$R_1\left(\frac{1}{3}\right) = \frac{1}{2!}\left(-2\frac{1+c^2}{(1-c^2)^2}\right)\left(\frac{1}{3}-0\right)^2 = -\frac{1}{9}\cdot\frac{1+c^2}{(1-c^2)^2}$$

<u>Note:</u> $\frac{1+c^2}{(1-c^2)^2}$ is <u>increasing</u> with $c \in [0, \frac{1}{3}]$ so

# Taylor remainder estimates

Def: $R_n(x) = f(x) - T_n(x)$ call this "error in the approx of $f(x)$ by $T_n(x)$" or "the $n$th remainder".

Philosophy: $R_n(x)$ "about" the next term (if things are working, i.e. $x$ is close enough to $a$)

"Exact" answer (Lagrange form):

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for some point $c$ between $a$ and $x$.

Warning: not ~~$\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$~~

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worksheet (4), (5)

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(5) continued:

$$\frac{1+0^2}{(1-0^2)^2} \le \frac{1+c^2}{(1-c^2)^2} \le \frac{1+\left(\frac{1}{3}\right)^2}{\left(1-\left(\frac{1}{3}\right)^2\right)^2}$$ i.e. $$1 \le \frac{1+c^2}{(1-c^2)^2} \le \frac{9\cdot 10}{8^2}$$

So (multiplying by $-\frac{1}{9}$)

$$-\frac{10}{64} \le R_1\left(\frac{1}{3}\right) \le -\frac{1}{9}$$

## 3. Higher order error estimates

> Let $R_n(x) = f(x) - T_n(x)$ be the *remainder*. Then there is $c$ between $a$ and $x$ such that
> $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

(6) Estimate the magnitude of the error in the quadratic approximation to $(4.1)^{3/2}$.

$f^{(3)}(x) = -\frac{3}{8}x^{-3/2}$ so by the Lagrange form of the remainder,

$$|R_2(4.1)| = \left|\frac{1}{3!}\cdot\left(-\frac{3}{8}c^{-3/2}\right)\cdot(4.1-4)^3\right| = \frac{1}{16{,}000}c^{-3/2}$$

for some $4 < c < 4.1$. Because $c^{-3/2}$ is decreasing on $[4, 4.1]$,

$c^{-3/2} \le 4^{-3/2} = \frac{1}{2^3} = \frac{1}{8}$ so $|R_3(4.1)| \le \frac{1}{16{,}000}\cdot\frac{1}{8} = \frac{1}{128{,}000}$.

(7) (Quiz, 2015) Consider a function $f$ such that $f^{(4)}(x) = \frac{\cos(x^2)}{3-x}$. Show that, when approximating $f(0.5)$ using its third-degree MacLaurin polynomial, the absolute value of the error is less than $\frac{1}{500}$.

By the Lagrange form of the remainder, the absolute value of the error is $|R_3(0.5)| = \frac{1}{4!}|f^{(4)}(c)|\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{24\cdot 16}\cdot\left|\frac{\cos(c^2)}{3-c}\right|$

for some $0 < c < \frac{1}{2}$. Now $|\cos(c^2)| \le 1$, $\frac{1}{|3-c|} \le \frac{1}{2\frac{1}{2}}$

so $|R_3(0.5)| \le \frac{1}{24\cdot 16}\cdot\frac{1}{2\frac{1}{2}} = \frac{1}{24\cdot 40} = \frac{1}{960} < \frac{1}{500}$.