11. Scientific applications (10/10/2019)

Goals.

(1) Velocity, speed, and acceleration.
(2) Other examples

Last Time.

Midterm cutoff: textbook §3.1 (= today's lecture)

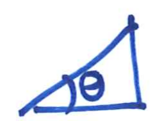
Midterm: Thursday 17th, in-class (half of class)

Implicit diff: if a function is defined implicitly (by equation) can still differentiate (by diff along the curve = diff the equation)

- use chain rule
- solve for y' in terms of x,y (always linear equation)

Inverse trig: $\arcsin(\sin\theta)$ can be computed by reduction to interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(\arccos x)$, $\mathrm{trig}(\mathrm{arctrig}(x))$ using right-angled triangle (choose two sides s.t. $\cos\theta = x$, then use Pythagoras to compute third side)



Derivatives: $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad (\arccos x)' = -(\arcsin x)'$

$(\arctan x)' = \frac{1}{1+x^2}$ $\quad$ (pf: inverse function rule)

Where do derivatives come from?

~~Have~~ Say we have quantity $f(t)$ depending on time.

- Average rate of change between times t_1, t_2 is:

$$\frac{f(t_2)-f(t_1)}{t_2-t_1}$$

- Instantaneous rate of change at time t_1 is

$$\lim_{t_2\to t_1} \frac{f(t_2)-f(t_1)}{t_2-t_1} = f'(t_1).$$

Today: Examples

Basic example: quantity is position of a particle
rate of change of position is called velocity.

$\Rightarrow$ If particle moving along line, at position $f(t)$ at time t, its velocity ~~at~~ at time t is defined to be $\frac{df}{dt}$.

(speed = magnitude of velocity)

Worksheet 1(a), (b)

Def: acceleration is the rate of change of the velocity.

hit breaks when $v<0$
then $a>0$, but we are slowing down

1(c)

Math 100 – WORKSHEET 11
SCIENTIFIC APPLICATIONS

1. Velocity and acceleration

(1) A particle's position is given by $f(t) = \frac{1}{\pi}\sin(\pi t)$.

(a) Find the velocity at time t, and specifically at $t = 3$.

The velocity is $v(t) = \frac{df}{dt} = \cos(\pi t)$, so $v(3) = \cos(3\pi) = \cos(\pi) = -1$

(at time 3 particle is moving leftward at speed 1)

(b) When is the particle moving to the right? to the left?

Particle is moving to the right at time $\bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2}+2\pi k, \frac{\pi}{2}+2\pi k\right)$

to the right $\bigcup_{k\in\mathbb{Z}} \left(-\frac{1}{2}+2k, \frac{1}{2}+2k\right)$

to the left $\bigcup_{k\in\mathbb{Z}} \left(\frac{1}{2}+2k, \frac{3}{2}+2k\right)$

(c) When is the particle accelerating? decelerating?

The acceleration is $a(t) = \frac{dv}{dt} = -\pi\sin(\pi t)$

the particle is accelerating when ($\cos(\pi t) > 0$ and $-\sin(\pi t) > 0$) $\leftarrow \left(-\frac{1}{2}, 0\right)$

or

$\left(\frac{1}{2}, 1\right) \rightarrow$ ($\cos(\pi t) < 0$ and $-\sin(\pi t) < 0$)

Date: 10/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

acceleration happens at times

$\left(\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, 1\right)\right) + 2\mathbb{Z}$

Basic example: constant acceleration.

If $a(t) = a$, $v(t) = at + v_0$ (constant deriv $\Rightarrow$ linear)

$\Rightarrow$ $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$ ($\frac{ds}{dt}$ = linear $\Rightarrow$ s = quadratic)

$\uparrow$ position

Worksheet (2), (3)

Lessons: - Can diff $s(t)$, $v(t)$, acc for positive/zero/negative values

- Can handle parameters / unknowns initial values

- distance travelled $\neq |s(t) - s(0)|$ unless velocity has constant sign

Can also diff other quantities

Worksheet

(2) (Final, 2016) An object is thrown straight up into the air at time $t = 0$ seconds. Its height in metres at time t seconds is given by $h(t) = s_0 + v_0 t - 5t^2$. In the first second the object rises by 5 metres. For how many seconds does the object rise before beginning to fall?

We are given that ~~$h(1) = 5$~~? no: $h(1) - h(0) = 5$ i.e.

$5 = (\cancel{s_0} + v_0 - 5) - \cancel{s_0}$ so $v_0 = 10\frac{m}{s}$

The object will keep rising until the velocity is 0.

$v(t) = \frac{dh}{dt} = v_0 - 10t = 10 - 10t$, so $\boxed{t = 1}$

(3) A emergency breaking car can decelerate at $9\frac{\mathrm{m}}{\mathrm{s}^2}$. How fast can a car drive so that it can come to a stop within 50m?

Say the car is driven at velocity v_0. After applying breaks, the velocity is $v(t) = v_0 - 9t$, in particular the car stops at time $t = \frac{v_0}{9}$. Position of car at time t is

$s(t) = v_0 t - \frac{1}{2} 9t^2$ (where breaking starts when car is at $s=0$)

So $s\left(\frac{v_0}{9}\right) = \frac{v_0^2}{9} - \frac{1}{2} 9 \frac{v_0^2}{9^2} = \frac{1}{2} \frac{v_0^2}{9}$ so stop in 50m if

$\frac{v_0^2}{18} = 50$ i.e. $v_0 = \sqrt{50 \cdot 18}\ \frac{m}{s} = \sqrt{100 \cdot 9}\ \frac{m}{s} = 30\ \frac{m}{s}$.

$= 30 \cdot 3.6\ \frac{km}{h} = 108\ \frac{km}{h}$

(Hw: add .2 sec of reaction time.)

2. OTHER APPLICATIONS

(1)

(a) Water is filling a cylindrical container of radius $r = 10$cm. Suppose that at time t seconds the height of the water is $(t + t^2)$ cm. How fast is the volume growing?

At time t, the water is occupying a cylindrical volume of radius r, height $h(t) = t + t^2$ (in cm), so volume $V(t) = \pi r^2 h(t)$.

Thus $$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} = 100\pi(1+2t)\ \frac{cm^3}{sec}$$

(b) A rocket is flying in space. The momentum of the rocket is given by the formula $p = mv$, where m is the mass and v is the velocity. At a time where the mass of the rocket is $m = 1000$kg and its velocity is $v = 500\frac{\text{m}}{\text{s}}$ the rocket is accelerating at the rate $a = 20\frac{\text{m}}{\text{s}^2}$ and losing mass at the rate $10\frac{\text{kg}}{\text{s}}$. Find the rate of change of the momentum with time.

At that time, $$\frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt}$$

$$= \left((-10)\cdot(500) + 1000\cdot 20\right)\cdot \frac{m\,kg}{sec^2}$$

$$= 15{,}000\ N$$

Example 6: Arrival process

Assumption: in a very short time Δt, prob student shows up is $r\Delta t$

What is the prob. that I wait t seconds for first arrival?

Idea: Say $P(t)$ = prob that nobody comes for time t

$$P(t+\Delta t) \approx P(t) \cdot (1 - r\Delta t)$$

$$P(t+\Delta t) - P(t) \approx -r\Delta t\, P(t)$$

$$\Rightarrow \frac{P(t+\Delta t) - P(t)}{\Delta t} \approx -rP(t)$$

take $\Delta t \to 0$ get, $P'(t) = -rP(t)$