4. The Intermediate Value Theorem; The Derivative (17/9/2019)

Goals.

(1) The Intermediate Value Theorem
   (a) With given endpoints
   (b) Free-form
(2) The derivative
   (a) Definition
   (b) Some calculations

Last Time.

Defined continuity: f cts at x_0 if:

$$\lim_{x\to x_0^-} f(x) = f(x_0) = \lim_{x\to x_0^+} f(x)$$

Automatically true if f def'd by formula (& formula makes sense)

+pictures of discontinuities

Worksheet (1), (3)

Math 100 – WORKSHEET 4
CONTINUITY: THE IVT; THE DERIVATIVE

1. Continuity

(1) Find c, d, e as appropriate such that each function is continuous on its domain:

$$f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ c & x = 1 \\ d - x^2 & x > 1 \end{cases}$$

On $[0,1)$, $(1,\infty)$ f is def'd by formula, hence cts.

At $x=1$, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$ so f is cts at 1 if

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (d-x^2) = d-1^2 = d-1$ $\quad 1 = c = d-1$

i.e. if $\boxed{c=1, d=2}$

$$\text{(Final 2013) } g(x) = \begin{cases} ex^2 + 3 & x \ge 1 \\ 2x^3 - e & x < 1 \end{cases}$$

g is cts on $(-\infty,1)$, $(1,\infty)$ (def'd by formula)

At $x=1$, $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (ex^2+3) = e\cdot 1^2 + 3 = g(1)$

$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (2x^2 - e) = 2-e$ so g is cts at $x=1$

if $e+3 = 2-e$, i.e. if $\boxed{e = -\frac{1}{2}}$

Date: 17/9/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Where are the following functions continuous?

$f(x) = \frac{1}{\sqrt{7-x^2}}$; $g(x) = \frac{x^2+2x+1}{2+\cos x}$; $h(x) = \frac{2+\cos x}{x^2+2x+1}$

(3) (Final 2011) Suppose f, g are continuous such that $g(3) = 2$ and $\lim_{x\to 3}(xf(x) + g(x)) = 1$. Find $f(3)$.

If f, g are cts, $\lim_{x\to3} f(x) = f(3)$, $\lim_{x\to3} g(x) = g(3)$

so

$\lim_{x\to3}(xf(x) + g(x)) = (\lim_{x\to3} x)(\lim_{x\to3} f(x)) + \lim_{x\to3} g(x) = 3f(3) + 2$

so $3f(3)+2=1$ so $f(3) =$

2. The Intermediate Value Theorem

Theorem. *Let $f(x)$ be continuous for $a \le x \le b$. Then $f(x)$ takes every value between $f(a), f(b)$.*

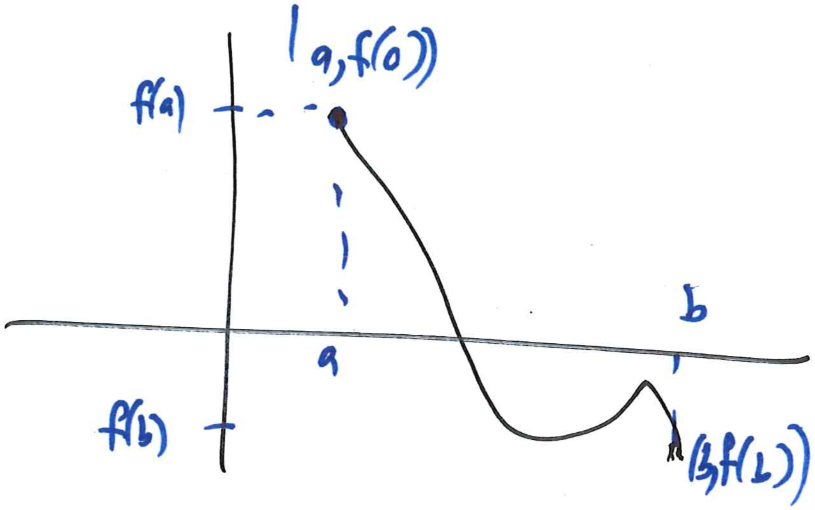
(1) Show that:

(a) $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \le x \le 1$.

$f(0) = 1$, $f(1) = -2$. Also, f is cts on $[0,1]$ (defined by formula), so by IVT, there is x in $[0,1]$ s.t. $f(x) = 0$

The intermediate value thm

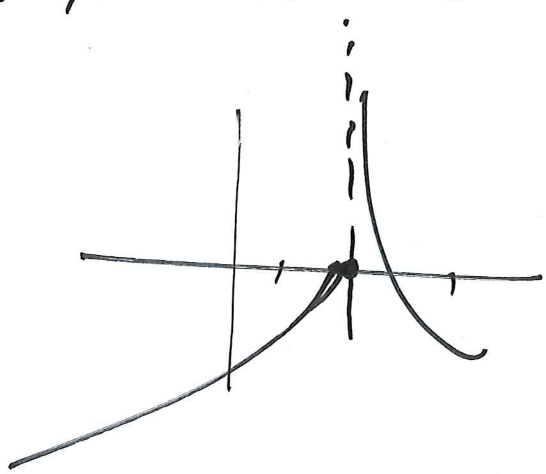
<u>Idea</u>: If f is cts on $[a, b]$, then f takes <u>every value</u> between $f(a), f(b)$



here $f(a) > 0$, $f(b) < 0$
want x_0 s.t. $f(x_0) = 0$

<u>Example</u>: Let $f(x) = e^x + x$. Show $f(x) = 0$ somewhere

worksheet (1)



Back to example: f is cts everywhere (def by formula)

$f(0) = e^0 + 0 = 1$, similarly $f(100) = e^{100} + 100$

$f(-100) = e^{-100} - 100 = \frac{1}{e^{100}} - 100$
$< 1 - 100 = -99$

so there is a zero between $[-100, 0]$.

hint: how would you start solving $x^2 = 2x+1$?

(b) $\sin x = x + 1$ has a solution.

Let ~~$f(x) = \sin x - (x+1)$~~ $f(x) = x+1-\sin x$. Want x_0 s.t. $f(x_0) = 0$

f is cts (defined by formula), $f(0) = 1$, $f(\frac{\pi}{2}) = \frac{\pi}{2}$,

$f(100) = 101 - \sin 100 \geq 100$

similarly $f(-\pi) = -\pi + 1 - \sin(-\pi) = -(\pi - 1) < 0$ | Since $f(100) > 0$, $f(-100) < 0$

$f(-100) = -99 - \sin(-100) \leq -98$ | by the IVT $f(x) = 0$ somewhere in between

(2) (Final 2011) Let $y = f(x)$ be continuous with domain $[0, 1]$ and range in $[3, 5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.

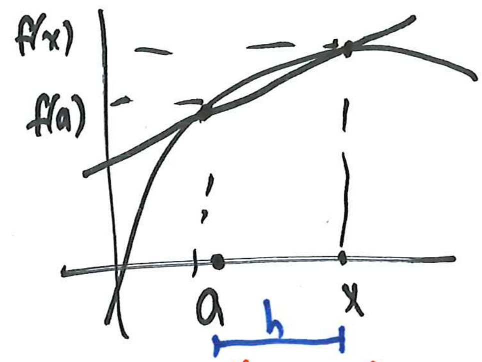
~~$f(x)$~~ Want x s.t. $f(x_0) = 2x_0 + 3$, ie x_0 s.t. $g(x) \overset{def}{=} f(x) - 2x - 3$ has $g(x_0) = 0$

such that

(3) (Final 2015) Show that the equation $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions.

The derivative

Recall from lecture 1: to find the slope of $y=f(x)$ at $x=a$ consider slopes of secant lines: $\dfrac{f(x)-f(a)}{x-a}$ and then take limit $x \to a$.



Def: If $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, say f is *differentiable* at $x=a$, call the value of the limit the *derivative* of f at a.

Equivalent: Write $x = a+h$ then $x \to a$ becomes $h \to 0$ so the derivative is also $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Example 1: Say $f(x) = 7x+3$, $a = 1$.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(7(1+h)+3)-(7+3)}{h} = \lim_{h\to 0} \frac{7h}{h} = 7.$$

(Notation: write $f'(a)$, or $\dfrac{df}{dx}(a)$, $\left.\dfrac{df}{dx}\right|_{x=a}$, ..

3. Definition of the Derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

(1) Find $f'(a)$ if

(a) $f(x) = x^2$, $a = 3$.

$$f'(3) = \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h}$$

$$= \lim_{h\to 0} (6+h) = 6$$

(b) $f(x) = \frac{1}{x}$, any a.

$$\lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{a-(a+h)}{(a+h)a}\right) = \lim_{h\to 0}\left(-\frac{1}{a(a+h)}\right) = -\frac{1}{a^2}$$