Lior Silberman's Math 535, Problem Set 1b: Analysis

Haar measure

Let X be a locally compact topological space. Write $C(X)$ for the space of continuous real-valued functions on X, and for $f \in C(X)$ write $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. It is well-known that the subspace $C_b(X) = \{f \in C(X) \mid \|f\|_\infty < \infty\}$ is complete in the supremum norm and that it contains the subspace $C_c(X)$ of compactly supported functions.

DEFINITION. A Radon *measure* on X is a linear functional $\mu : C_c(X) \to \mathbb{C}$ such that $\mu(f) \geq 0$ if $f \geq 0$ (that is, if $f(x) \in \mathbb{R}_{\geq 0}$ for each x). If μ is a Radon measure and $f \in C_c(X)$ we often write $\int f \, d\mu$ instead of $\mu(f)$.

1. (Preliminatires)
   (a) Show that the closure of $C_c(X)$ in $C_b(x)$ is the space $C_0(X)$ of functions vanishing at infinity (continuous functions f such that for all $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ if $x \notin K$.
   (b) Let $X' \subset X$ and let μ be a Radon measure on X. Show that $\mu \upharpoonright_{C_c(X')}$ is a Radon measure on X'.
   (b) In particular, suppose Y is compact. Show that a Radon measure on Y is a bounded linear functional on $C(Y) = C_b(Y) = C_c(Y)$.

2. Let G be a locally compact topological group.
   (a) Let $f, f' \in C_c(G)$ be non-negative, and let $U \subset G$ be open. Set
   $$(f : U) = \inf\left\{\sum_{i=1}^{n} \alpha_i \mid \alpha_i \geq 0, f \leq \sum_{i=1}^{n} \alpha_i \cdot 1_{g_i U}\right\}.$$
   Show that $0 \leq (f : U) < \infty$. Assuming $f' \neq 0$ show that $(f : U) \leq (f' : U)(f : f')$ for an appropriately defined $(f : f')$ which is independent of U.
   (b) Let $\mathcal{N}$ be the set of open neighbourhoods of the identity in G; for $U \in \mathcal{N}$ set $F_U = \{V \in \mathcal{N} \mid V \subset U\}$. Show that $\mathcal{F} = \{S \subset \mathcal{N} \mid \exists U : S \supset F_U\}$ is a filter on $\mathcal{N}$ (that is, if $S_1, S_2 \in \mathcal{F}$ and $T \subset \mathcal{N}$ then $S_1 \cap S_2, S_1 \cup T \in \mathcal{F}$). Show that for any $V \in \mathcal{N}$ there is $S \in \mathcal{F}$ with $V \notin S$ ("$\mathcal{F}$ is not contained in any principal filter"). Let $\omega \subset \mathcal{N}$ be a maximal filter containing $\mathcal{F}$.
   (c) Fix $f_0 \in C_c(G)$ which is non-negative and non-zero. Show that $\mu(f) \stackrel{\text{def}}{=} \lim_{U \to \omega} \frac{(f : U)}{(f_0 : U)}$ extends to a G-invariant Radon measure on G. Such μ is called a (left) *Haar measure* on G.
   (d) Show that $\mu(f) > 0$ for all non-negative non-zero $f \in C_c(G)$.
   (e) Suppose G is non-compact. Show that μ is an *infinite measure*: that $\mu : C_c(X) \to \mathbb{C}$ is unbounded with respect to the supremum norm.

3. (Uniqueness of Haar measure) Let G be a locally compact topological group and let μ_1, μ_2 be a left Haar measure on G.
   (a) Given $f \in C_c(G)$ show that f is *uniformly continuous:* for any $\varepsilon > 0$ there is an open subset U such that for all $x \in G$, $u \in U$ we have$|f(xu) - f(x)| < \varepsilon$. Furthermore, we can choose U so that $\mathrm{supp}(f)U$ is contained in any fixed compact set K.

(b) Let $\chi \in C_c(U)$ be positive such that $\mu(\chi) = 1$ and let $(f \star \chi)(x) = \int_G f(xu)\chi(u)\,d\mu_1(u)$. Show that $\|f \star \chi - f\|_\infty \le \varepsilon$ and hence

$$\left| \int d\mu_2(x) \int d\mu_1(u) f(xu)\chi(u) - \int d\mu_2(x) f(x) \right| \le \varepsilon \mu_2(K)\,.$$

(c) Changing variables on the LHS show that

$$\left| \int d\mu_2(x) f(x) - E \int d\mu_1(x) f(x) \right| \le \varepsilon \mu_2(K)$$

with $E = \int \chi(x^{-1}) d\mu_2(x) > 0$.

(d) For any $f, g \in C_c(G)$ show that

$$|\mu_1(g)\mu_2(f) - \mu_1(f)\mu_2(g)| = 0$$

and hence that μ_1 and μ_2 are proportional.

4 Fix a left Haar measure *mu*.

(a) For $f \in C_c(G)$ and $g \in G$ let $(R_g f)(x) = f(xg)$ be the left regular representation. Show that $\mu_g(f) \stackrel{\text{def}}{=} \mu(R_g f)$ is also a left Haar measure on G. It follows that there is $\delta_G(g) \in \mathbb{R}^\times_{>0}$ such that $\mu_g(f) = \delta_G(g^{-1})\mu(f)$ for all f.

RMK The g^{-1} is there so that $\mu(Ag) = \delta_G(g)\mu(A)$ for every left Haar measure μ, measurable $A \subset G$ and $g \in G$.

(b) Show that $\delta_G \colon : G \to \mathbb{R}^\times_{>0}$ is a continuous group homomorphism.

DEF The map $\delta_G \colon G \to \mathbb{R}^\times_{>0}$ is called the *modular character* of G. The group G is called *unimodular* if δ_G is the trivial character (identically 1).

(c) Show that $\mu\left(f(x^{-1})\delta(x)\right)$ is a right Haar measure on G. Conclude that G is unimodular if every left Haar measure is a right Haar measure.

(d) Suppose G is compact. Show that $\mathrm{Hom}_{\mathrm{cts}}\left(G, \mathbb{R}^\times_{>0}\right) = \{1\}$ and conclude that G is unimodular.

(e) Show that every abelian group and every discrete group is unimodular.

5. (Example of Haar measure) Let $\mathrm{GL}_n(\mathbb{R}) = \{g \in M_n(\mathbb{R}) \mid \det g \neq 0\}$. Let μ be the measure on $\mathrm{GL}_n(\mathbb{R})$ with density $\frac{1}{|\det(g)|^n}$ wrt Lebesgue measure – in other words:

$$\int f(g)\,d\mu(g) = \iint f\left((g_{ij})_{i,j=1}^n\right) \frac{1}{|\det(g)|^n} dg_{11} \cdots dg_{nn}\,.$$

Show that μ is a left- and right-invariant Haar measure.

Supplement: Tensor products of locally convex vector spaces

Let X, Y be Banach spaces and let $X \otimes Y$ be their *algebraic* tensor product.

6. A *cross norm* on $X \otimes Y$ is a norm such that

$$\begin{aligned} \forall x \in X, y \in Y &\quad : \quad \|x \otimes y\| = \|x\|_X \|y\|_Y \\ \forall x' \in X', y' \in Y' &\quad : \quad \|x' \otimes y'\| = \|x'\|_{X'} \|y'\|_{Y'} \end{aligned}$$

(a) Show that $\|t\|_\pi = \inf\{\sum_{i=1}^r \|x_i\|_X \|y_i\|_Y \mid t = \sum_{i=1}^r x_i \otimes y_i\}$ defines a norm on $X \otimes Y$, and that $\|t\|_\pi \ge \|t\|$ for all cross norms $\|\cdot\|$.

(b) Show that $\|t\|_{\varepsilon} = \sup\{|(x' \otimes y')(t)| \, x' \in X', y' \in Y, \|x'\|_{X'} = \|y'\|_{Y'} = 1\}$ defines a norm on $X \otimes Y$, and that $\|t\|_{\varepsilon} \leq \|t\|$ for all cross norms $\|\cdot\|$.

(c) Let $X \otimes_{\varepsilon} Y, X \otimes_{\pi} Y$ be the completions of $X \otimes Y$ with respect to these norms. Obtain a continuous inclusion $X \otimes_{\varepsilon} Y \hookrightarrow X \otimes_{\pi} Y$.

RMK In general this is not an isomorphism.