Math 322, lecture 21 , 21/11/2017

Today: (1) PS9 (2) ~~Solvable~~ Nilpotence

PS9 , 2(a)

Let G have order $255 = 3 \cdot 5 \cdot 17$.

Sylow theory: $P_3 \simeq C_3$, $P_5 \simeq C_5$, $P_{17} \simeq C_{17}$,

$n_3(G) \in \{1, 5 \cdot 17\}$, $n_5(G) = \{1, 3 \cdot 17\}$, $n_{17}(G) = \{1\}$

divide $5 \cdot 17$
be $\equiv 1\ (3)$

(a) $n_{17}(G) = 1$ so P_{17} is normal. Then $g \cdot x = gxg^{-1}$ defines an action of G on P_{17} by automorphisms, i.e. a hom $G \to \mathrm{Aut}(P_{17}) \simeq \mathrm{Aut}(C_{17}) \simeq (\mathbb{Z}/17\mathbb{Z})^\times \simeq C_{16}$

fact from PS — fact from class

Now order of image of this hom divides $\# G$ (first isom thm) and divides 16 (Lagrange's thm in $\mathrm{Aut}(P_{17})$).

So the image is trivial, i.e. the action is trivial: $gxg^{-1} = x$ for all $g \in G$, $x \in P_{17}$, i.e. $P_{17} \subseteq Z(G)$

(b) now no "way" to have orbit of size containing 17 in the action of G on itself by conjugation.

Now ~~$\bullet$~~ $\mathrm{Syl}_5(G)$ is a single conjugacy class, so by orbit-stabilizer thm $n_5(G) = [G : N_G(P_5)]$

$$\begin{array}{c} G \\ | \; \text{at most } 3 \\ N_G(P_5) \\ | \; \text{includes } 17 \\ P_5 \\ | \; 5 \\ \{e\} \end{array}$$

Now P_{17} is central, so acts trivially on $Syl_5(G)$, hence $P_{17} \subseteq N_G(P_5)$

so $\# N_G(P_5)$ is divisible by 5 ($\# P_5$) and 17 ($\# P_{17}$)

so $[G : N_G(P_5)]$ divides 3.

but $n_5(G)$ not 3, so $n_5(G) = 1$

⋮

Let G have order $140 = 2^2 \cdot 5 \cdot 7$

Then $\# P_2 = 4$, $\# P_5 = 5$, $\# P_7 = 7$

$n_2(G) \in \{1, 5, 7, 35\}$, $n_5(G) \in \{1\}$, $n_7(G) \in \{1\}$

$\Rightarrow$ the subgps P_5, P_7 are normal, disjoint (relatively prime orders)

$\Rightarrow$ $H = P_5 P_7$ is a subgp.

Since P_5, P_7 are <u>both</u> normal, disjoint, they commute

(HW: if $x \in P_5$, $y \in P_7$ then $[x, y] = x y x^{-1} y^{-1} \in P_5 \cap P_7 = \{e\}$)

<u>or</u>: H is a gp of order $35 = 5 \cdot 7$. Since $7 \not\equiv 1 \ (5)$, $H \cong C_{35}$.

(thm on pq-groups)

H is normal (generated by normal subgp)
disjoint from P_2 (order of H is odd).
So $P_2 H$ is a semidirect pdt of order $4 \cdot 35 = 140$.
I.e. $G \cong P_2 \rtimes H. \cong P_2 \rtimes C_{35}$

<u>Remains</u>: (1) Study $\mathrm{Hom}(P_2, \mathrm{Aut}(C_{35}))$ up to automorphism
(2) check for non-isom

For this: $\mathrm{Aut}(C_{35}) = (\mathbb{Z}/35\mathbb{Z})^\times \underset{\substack{\uparrow \\ \text{CRT}}}{\cong} (\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z})^\times$ respects mult.

$\cong (\mathbb{Z}/5\mathbb{Z})^\times \times (\mathbb{Z}/7\mathbb{Z})^\times \cong C_4 \times C_6$
$\cong C_4 \times C_2 \times C_3$

<u>PS8, Problem 6</u>: up to isom, $C_4 \rtimes C_{35}$ determined
by subgp of $C_4 \times C_6$ of order 14.

subgp of of order 4, generated by $([1],[0])$, $([1],[3])$
of $C_4 \times C_6$ one of

Get two subgps: $C_4 \times \{[0]\}$, $\{([0],[0]), ([1]_4,[3]_6), ([2]_4,[0]_6), ([3],[3])\}$

~~These give~~ The resulting semidirect
pdts are distinct: in first case, C_4 only acts on P_5,
commutes with P_7,
$C_4 \rtimes_{\varphi_1} C_{35}$, $C_4 \rtimes_{\varphi_2} C_{35}$ | In the second case no

(first case is $(C_4 \rtimes_{\varphi_1} C_5) \times C_7$

subgp of order 2: same as elements of order 2, got:

$$([2]_4, [0]_6), \quad ([0]_4, [3]_6), \quad ([2]_4, [3]_6)$$

(mean: generator of $P_2 \cong C_4$ will act by one of those)

They are distinct: in first two case P_2 will commute with P_7 or P_5, in second with neither

subgp of order 1: can have $C_4 \times C_{35} \cong C_{140}$

Point: let ~~G, G'~~ G, G' be gps of order 140

~~say P_2, P_2' (2-Sylow subgps) are both~~

let $f: G \to G'$ be an isom

Saw: G, G' have unique subgps of order 35, H, H', so $f(H) = H'$. Let $P_2 < G$ be a 2-Sylow subgp, then $f(P_2) = P_2'$ is a 2-Sylow subgp (also has order 4)

let $\alpha: P_2 \to \mathrm{Aut}(H)$ be the conjugation action

$\alpha': P_2' \to \mathrm{Aut}(H')$ " " " "

Then $\alpha' = f_* \circ \alpha \circ f^{-1}$ where if $\beta \in \mathrm{Aut}(H)$, $f_* \beta = f \circ \beta \circ f^{-1}$

in particular, conjugation by $f|_H$ (as a map $\mathrm{Aut}(H) \to \mathrm{Aut}(H')$) gives isom $\alpha(P_2) \cong \alpha'(P_2')$

Nilpotence

Motivation: Problem: Given $f \in \mathbb{Q}[x]$, find roots of f.

(eg: $f(x) = x^2 + bx + c$, roots are $\frac{-b \pm \sqrt{b^2 - 4c}}{2}$)

(similar if $\deg f = 3$, or 4.)

Assume f irreducible = no factors in $\mathbb{Q}[x]$ other than $1, f$.

Galois: Let $\alpha_1, .., \alpha_d \in \mathbb{C}$, $d = \deg f$ be the roots

Let F = smallest field of complex numbers containing $\alpha_1, .., \alpha_d$.

$= \mathrm{Span}_{\mathbb{Q}} \{1, \alpha_i, \alpha_i\alpha_j, \alpha_i\alpha_j\alpha_k, \dots\}$

Fact: $\dim_{\mathbb{Q}} F \leq d!$

(example: $f(x) = x^2 + 1$, $F = \mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$

$f(x) = x^3 - 2$, $F = \mathbb{Q}(\sqrt[3]{2}, \omega) = \mathrm{Span}\{\sqrt[3]{2}^i \omega^j\}_{\substack{i = 0,1,2 \\ j = 0,1}}$)

$\omega = -\frac{1}{2} + \sqrt{\frac{3}{2}}\, i$ $b \neq 0$

Set $G = \mathrm{Gal}(f) = \{\sigma \in S_d \mid$ permuting $\alpha_1, .., \alpha_d$ respects multiplication, addition$\}$

(eg. if $\alpha_1\alpha_2 + \alpha_3\alpha_4 = 0$

then must have $\alpha_{\sigma(1)}\alpha_{\sigma(2)} + \alpha_{\sigma(3)}\alpha_{\sigma(4)} = 0$)

Thm: (Abel) ~1800 Suppose $\mathrm{Gal}(f)$ is commutative. Then can express roots of f using $+, \cdot, -, \div, \sqrt[k]{\ }$

(hence commutative groups called "Abelian")

Thm (Galois, ~1830): Explicit group-theoretic condition ("solvability") s.t. roots of f expressible by radicals iff Gal(f) is solvable.

Also, S_2, S_3, S_4 solvable $\Rightarrow$ quadratic, cubic, quartic formulas

S_n if $n \geq 5$ not solvable (because A_n simple if $n \geq 5$)

so no quintic formula