Lior Silberman's Math 312: Problem Set 4 (due 6/6/18)

Multiplicative Order

1. Let n be a pseudoprime to base 2 (recall that this means $2^{n-1} \equiv 1\,(n)$). Show that $m = 2^n - 1$ is also a pseudoprime to base 2.
   Hint: Show that $n|m-1$ and use the fact that you know the order of 2 mod m.

*2. Let p be a prime divisor of the nth Fermat number $F_n = 2^{2^n} + 1$.
   (a) Find the order of 2 mod p.
   (b) Show that $p \equiv 1\,(2^{n+1})$.
   (c) Show that for any $a \geq 1$ there are infinitely many primes p for which the order of 2 mod p is divisible by 2^a.
   RMK Note that (b) simplifies the search for prime divisors of Fermat numbers. We will later show that $p \equiv 1\,(2^{n+2})$ holds.

3. Elements of order 2 mod m.
   (a) Let p be odd, and let $k \geq 1$. Show that the congruence $x^2 \equiv 1\,(p^k)$ has only the two obvious solutions $x \equiv \pm 1\,(p^k)$.
   Hint: Can both $x-1, x+1$ be powers of p?
   (*b) Let n be an odd number, divisible by exactly r distinct primes. Set up a bijection between congruence classes mod n satisfying $x^2 \equiv 1\,(n)$ and functions $f \in \{\pm 1\}^r$. Conclude that there are precisely 2^r congruence classes mod n which solve the equation.

4. Using Fermat's Little Theorem, show that for all integers n, $30|n^9 - n$.
   Hint: For each prime $p|30$ show that $n^p - n|n^9 - n$ as polynomials.

Wilson's Theorem

5. We will show that if $n \geq 6$ is composite then $(n-1)! \equiv 0\,(n)$.
   (a) (The easy case) Assume first that n is divisible by at least two distinct primes, that is that $n = \prod_{j=1}^{r} p_j^{k_j}$ for some distinct primes p_j where $k_j \geq 1$ for all j and $r \geq 2$. Show that $(n-1)! \equiv 0\,(n)$.
   Hint: It is enough to show the congruence mod each $p_j^{k_j}$ separately. Why is $(n-1)!$ divisible by $p_j^{k_j}$?
   (b) Let p be prime and let $k \geq 3$. Show that $p^k|(p^k-1)!$
   Hint: Find some powers of p dividing the factorial.
   (c) Let $p \geq 3$ be prime. Show that $p^2|(p^2-1)!$
   Hint: Now you need to consider multiples of p as well.
   RMK Note that $3! \not\equiv 0\,(4)$. Ensure that your solution to (c) used the fact that $p \neq 2$ at some point!

The Euler Function and RSA

Recall that $\phi(m) = \#\{1 \le a \le m \mid (a,m) = 1\}$, and that for p prime $\phi(p) = p-1$.

6. Explicit calculations.
   (a) Calculate $\phi(4)$, $\phi(9)$, $\phi(12)$, $\phi(15)$.
   (b) Show that $\phi(12) = \phi(3)\phi(4)$ and $\phi(15) = \phi(3)\phi(5)$ but that $\phi(4) \neq \phi(2)\cdot\phi(2)$, $\phi(9) \neq \phi(3)\cdot\phi(3)$.

7. Let p,q be distinct primes and let $m = pq$.
   (a) Show that there are $p+q-1$ integers $1 \le a \le m$ which are not relatively prime to m.
   Hint: What are the possible values of $\gcd(a,m)$? For which a do they occur?
   (b) Show that $\phi(pq) = (p-1)(q-1)$.
   RMK This means in particular that $\phi(pq) = \phi(p)\phi(q)$.
   (c) Give a formula for $p+q$ in terms of $m, \phi(m)$.
   SUPP Show how to factor m given $m, \phi(m)$.

8. Fix an integer m and two positive integers d,e so that $de \equiv 1\,(\phi(m))$. Define functions E,D by $E(x) = x^e \mod m$ and $D(y) = y^d \mod m$ (in other words, raise to the appropriate power and keep remainder mod m).
   (a) Let $M = \{1 \le a \le m \mid (a,m) = 1\}$ be the set of invertible residues ($\phi(m)$ is the size of this set). Show that both D,E map the set M into itself.
   (b) Show that for any $x,y \in M$, $D(E(x)) = x$ and $E(D(y)) = y$.
   Hint: Euler's Theorem.

Supplementary problems (not for submission)

A. (The binomial formula) Prove by induction on $n \ge 0$ that for all x,y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

B. Let p be an odd prime.
   (a) Show that $(p-1)! \equiv (-1)^{\frac{p-1}{2}} \left(\left(\frac{p-1}{2}\right)!\right)^2 (p)$. Conclude that if $p \equiv 1\,(4)$ then there is $a \in \mathbb{Z}$ such that $a^2 \equiv -1\,(p)$.
   (b) Conversely, assume that $a^2 \equiv -1\,(p)$ for some integer a. Show that the order of a mod p is exactly 4 and conclude that $p \equiv 1\,(4)$.

C. Let p be a prime and let $0 \le k < p$. Show that $\binom{p-1}{k} \equiv (-1)^k\,(p)$.