Math 101 – SOLUTIONS TO WORKSHEET 28
ABSOLUTE CONVERGENCE

1. More Tail Estimates

(1) It is known that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
   (a) How close is $\frac{1}{2} - \frac{1}{6} + \frac{1}{24}$ to $\frac{1}{e}$?
   (b) How many terms are needed to approximate $\frac{1}{e}$ to within $\frac{1}{1000}$?

   Solution: The series $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ is alternating, and $n!$ is increasing to infinity so that $\frac{1}{n!}$ monotonically decrease to zero. By the alternating series test, the error is bounded by the next term.

   (a) The next term after $\frac{1}{24} = \frac{1}{4!}$ is $-\frac{1}{5!} = \frac{1}{120}$ so

$$\left|\frac{1}{e} - \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)\right| \leq \frac{1}{120}\,.$$

   (b) If we want to approximate $\frac{1}{e}$ to within $\frac{1}{1000}$ we need to keep terms until one is smaller than than. We have $\frac{1}{6!} = \frac{1}{720}$ and $-\frac{1}{7!} = -\frac{1}{5040}$ so keeping the first seven terms we have

$$\left|\frac{1}{e} - \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)\right| \leq \frac{1}{5040} < \frac{1}{1000}\,.$$

(2) The *error function* is (roughly) given by $\mathrm{erf}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$. How many terms are needed to approximate $\mathrm{erf}(\frac{1}{10})$ to within 10^{-11}?

   Solution: Using $x = \frac{1}{10}$ gives the series

$$\mathrm{erf}\left(\frac{1}{10}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)10^{2n+1}}\,.$$

   Since each of the factors of $n!(2n+1)10^{2n+1}$ is increasing, the terms of the series terms are monotonically decreasing in magnitude, tending to zero, and are clearly alternating in sign. For $n = 4$ we have $n!(2n+1)10^{2n+1} = 24 \cdot 9 \cdot 10^9 > 100 \cdot 10^9 = 10^{11}$ since $24 \cdot 9 > 20 \cdot 5 = 100$. By the alternating series test taking the first four terms is sufficient:

$$\left|\mathrm{erf}\left(\frac{1}{10}\right) - \left(1 - \frac{1}{300} + \frac{1}{10^4} - \frac{1}{42 \cdot 10^7}\right)\right| < 10^{-11}\,.$$

2. Absolute Convergence

(3) Decide if each sequence/series converges:

$$\square\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty} \quad \square\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \quad \square\left\{\frac{(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty} \quad \square\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

   Solution: $\lim_{n\to 1} \frac{1}{\sqrt{n}} = 0$, so also $\lim_{n\to\infty} \frac{-1}{\sqrt{n}} = 0$, and by the squeeze theorem $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$, so both sequences *converge*. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$ so it *diverges* while the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ *converges* by the alternating series test.

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(4) Place checkmarks

	Converges		Diverges
	Absolutely	Conditionally	
$\sum_{n=1}^{\infty}(-1)^n$			X
$\sum_{n=1}^{\infty}\frac{1}{n^2}$	X		
$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$	X		
$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$		X	
$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$	X		
(*)$\sum_{n=1}^{\infty}\frac{\sin n}{n}$		X	

Solution: * The series $\sum_{n=1}^{\infty}(-1)^n$ diverges, for example by the nth element test – the terms are either $+1, -1$ and in any case don't tend to zero.
* The positive series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series with $p = 2 > 1$). The series is also absolutely convergent because each term is equal to its own absolute value.
* In the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, replacing each term by its absolute value gives the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is convergent (see above) so this series is also absolutely convergent.
* Replacing each term in of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ by its absolute value gives the series $\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges (it's the harmonic series), so $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is not absolutely convergent. But the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ does converge by the alternating series test: its terms alternate in sign, decrease in magnitude, and tend to zero. It follows that the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges conditionally.
* Replacing each term in of the series $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ by its absolute value gives the positive series $\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2}$ which coverges by comparison with the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (we have $0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$ for all n).
* The example $\sum_{n=1}^{\infty}\frac{\sin n}{n}$ is just for flavour – properly dealing with it is beyond the level of Math 101. The basic idea is that as n varies, the angle "n radians" looks like a random angle around the circle. this makes the numbers $\sin n$ be distributed in $[-1, 1]$ according to the sign curve. First, replacing each term with its absolute value gives the series $\sum_{n=1}^{\infty}\frac{|\sin n|}{n}$ and since the values $\sin n$ are random, they aren't often close to zero, and you can roughly compare our series to $\sum_{n=1}^{\infty}\frac{1/10}{n}$ which diverges. On the other hand, without absolute values there is a lot of cancellation between the terms (to see the cancellation note that $\int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = 0$ and that $\left|\int_0^T \sin\theta\,\mathrm{d}\theta\right| \le 2$ no matter how big T is) and this makes the series $\sum_{n=1}^{\infty}\frac{\sin n}{n}$ converge.

(*) beyond the scope of Math 101

3. Ratio test

(5) Decide whether the following series converge:

(a) $\sum_{n=0}^{\infty}\frac{n}{2^n}$

Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}}\Big/\frac{n}{2^n} = \frac{n+1}{n}\cdot\frac{2^n}{2^{n+1}} = \frac{1}{2}\left(1+\frac{1}{n}\right) \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$ so the series converges by the ratio test.

(b) $\sum_{n=0}^{\infty}\frac{n!}{2^n}$

Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{2^{n+1}}\Big/\frac{n!}{2^n} = \frac{(n+1)!}{n!}\cdot\frac{2^n}{2^{n+1}} = \frac{n+1}{2} \xrightarrow[n\to\infty]{} \infty > 1$ so the series diverges by the ratio test.

(c) $\sum_{n=0}^{\infty}\frac{2^n}{n!}$

Solution: We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$ so the series converges by the ratio test.

(d) For which values of x does $\sum_{n=0}^{\infty} nx^n$ converge?

Solution: Let $a_n = nx^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x|^{n+1}}{n\,|x|^n} = \left(1+\frac{1}{n}\right)|x| \xrightarrow[n\to\infty]{} |x|\ .$$

By the ratio test, the series *converges* if$|x| < 1$ and *diverges* if $|x| > 1$. If $|x| = 1$ then $|a_n| = n\,|x|^n = n \xrightarrow[n\to\infty]{} \infty$ so the series *diverges* by the divergence test. We conclude that the series converges exactly when $|x| < 1$, that is for $x \in (-1, 1)$.