Math 101 – SOLUTIONS TO WORKSHEET 23
SERIES

1. Tool: Squeeze Theorem

(1) Determine if each sequence is convergent or divergent. If convergent, evaluate the limit.

(a) (Final 2013) $\left\{(-1)^n \sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$.

Solution: For $n \geq 1$, $\sin\left(\frac{1}{n}\right) \geq 0$ so

$$-\sin\left(\frac{1}{n}\right) \leq (-1)^n \sin\left(\frac{1}{n}\right) \leq \sin\left(\frac{1}{n}\right).$$

We have seen in 1(d) that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$ and it follows that $\lim_{n\to\infty}\left(-\sin\left(\frac{1}{n}\right)\right) = -\lim_{n\to\infty}\sin\left(\frac{1}{n}\right) = 0$ as well. By the squeeze theorem we conclude that

$$\lim_{n\to\infty}(-1)^n \sin\left(\frac{1}{n}\right) = 0.$$

(b) (Final 2011) $\left\{\frac{\sin(n)}{\log(n)}\right\}_{n=2}^{\infty}$ (why do we have $n \geq 2$ here?)

Solution: Since $\lim_{n\to\infty}\log(n) = \lim_{x\to\infty}\log(x) = \infty$, we have $\lim_{n\to\infty}\frac{1}{\log n} = 0$. Also, for every n we have $-1 \leq \sin n \leq 1$ so that

$$-\frac{1}{\log n} \leq \frac{\sin n}{\log n} \leq \frac{1}{\log n}.$$

Since $\lim_{n\to\infty} -\frac{1}{\log n} = -\lim_{n\to\infty}\frac{1}{\log n} = 0$ also, we have by the squeeze theorem that

$$\lim_{n\to\infty}\frac{\sin n}{\log n} = 0.$$

(c) (Math 105 Final 2012) $a_n = 1 + \frac{n!\sin(n^3)}{(n+1)!}$.

Solution: We have $(n+1)! = n!(n+1)$ so $a_n = 1 + \frac{\sin(n^3)}{n+1}$, and for every n we have $-1 \leq \sin(n^3) \leq 1$ so that

$$1 - \frac{1}{n+1} \leq 1 + \frac{\sin(n^3)}{n+1} \leq 1 + \frac{1}{n+1}.$$

Now $\lim_{n\to\infty}\left(1 \pm \frac{1}{n+1}\right) = 1 \pm \lim_{x\to\infty}\frac{1}{x} = 1$ and it follows from the squeeze theorem that

$$\lim_{n\to\infty} 1 + \frac{n!\sin(n^3)}{(n+1)!} = 1.$$

2. Skill 1: Geometric series and decimal expansions

(1) (Final 2013) Find the sum of the series $\sum_{n=2}^{\infty}\frac{3\cdot 4^{n+1}}{8\cdot 5^n}$. Simplify your answer.

Solution: We write this as $\sum_{n=2}^{\infty}\frac{12}{8}\left(\frac{4}{5}\right)^n$ so this is a geometric series with ratio $\frac{4}{5}$ and first term $\frac{3}{2}\left(\frac{4}{5}\right)^2$. Its sum is therefore

$$\frac{3}{2}\frac{(4/5)^2}{1-\frac{4}{5}} = \frac{3\cdot 16}{2\cdot 5\cdot 5\cdot(1-\frac{4}{5})} = \frac{24}{5\cdot(5-4)} = \boxed{\frac{24}{5}}.$$

(2) Express each decimal expansion using a geometric series, sum the series, then simplify to obtain a rational number.

Date: 6/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(a) $0.333333\ldots$

Solution: We have $0.33333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = \frac{3}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{3}{9} = \boxed{\frac{1}{3}}$.

(b) $0.5757575757\ldots$

Solution: This is $\frac{57}{100} + \frac{57}{(100)^2} + \frac{57}{(100)^3} + \cdots = \frac{57}{100} \cdot \frac{1}{1-\frac{1}{100}} = \boxed{\frac{57}{99}}$.

(c) $0.6545454545454\ldots$

Solution: Here we have to be more careful:

$$\begin{aligned} 0.6545454545454\ldots &= 0.6 + \frac{54}{1000} + \frac{54}{100,000} + \frac{54}{10,000,000} + \cdots = 0.6 + \frac{54}{1000}\left(1 + \frac{1}{100} + \frac{1}{(100)^2} + \frac{1}{(100)^3} + \cdots\right) \\ &= 0.6 + \frac{54}{1000} \cdot \frac{1}{1-\frac{1}{100}} = \frac{6}{10} + \frac{54}{10 \cdot 99} = \frac{3}{5} + \frac{3}{5 \cdot 11} = \frac{3 \cdot 12}{5 \cdot 11} = \boxed{\frac{36}{55}}. \end{aligned}$$

3. Skill 2: Telescoping series

(3) Write an expression for the partial sums, decide if the series converges, and if so determine the sum.

(a) (Final 2015) $\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right)$

Solution: The Nth partial sum is

$$\left(\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{4}\right)\right) + \left(\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{6}\right)\right) + \left(\cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{8}\right)\right) + \cdots + \left(\cos\left(\frac{\pi}{N}\right) - \cos\left(\frac{\pi}{N+1}\right)\right)$$

where every cosine cancels except for the first and the last giving us

$$s_N = \cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{N+1}\right).$$

This converges as $N \to \infty$ with

$$\lim_{N\to\infty} s_N = \cos\left(\frac{\pi}{3}\right) - \cos(0) = \frac{1}{2} - 1 = \boxed{-\frac{3}{2}}.$$

(b) $\sum_{n=1}^{\infty}\left(n^2 - (n+1)^2\right)$

Solution: The nth partial sum is $\left(1^2 - 2^2\right) + \left(2^2 - 3^2\right) + \cdots + \left(n^2 - (n+1)^2\right) = 1^2 - (n+1)^2$ and these clearly tend to $-\infty$ as $n \to \infty$ so the series diverges.

(c) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

Solution: We have $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$ (partial fractions). Writing the partial sum

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

we see that every fraction appears twice (with opposite signs) except for $1, \frac{1}{2}, -\frac{1}{n+1}, -\frac{1}{n+2}$ so

$$s_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus

$$\lim_{n\to\infty} s_n = \frac{3}{2} - 0 - 0 = \boxed{\frac{3}{2}}$$

and the series converges to $\frac{3}{2}$.

(d) $\sum_{n=0}^{\infty}(\arctan(n) - \arctan(n+1))$

Solution: The nth partial sum is

$$\begin{aligned} (\arctan(0) - \arctan(1)) + (\arctan(1) - \arctan(2)) + \cdots + (\arctan(n-1) - \arctan(n)) &= \arctan(0) - \arctan(n) \\ &= -\arctan(n). \end{aligned}$$

Now $\lim_{n\to\infty} \arctan(n) = \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$ so the series converges to $\boxed{-\frac{\pi}{2}}$.