Math 101 – SOLUTIONS TO WORKSHEET 22
SEQUENCES

1. Skill 1: expressions for sequences

(1) For each of the following sequences, write a formula for the general term
  (a) $\{1, 2, 3, 4, 5, 6, \cdots\}$
  Solution: $a_n = n$.
  (b) $\left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \cdots\right\}$
  Solution: $a_n = \frac{1}{n^2}$
  (c) $\{3, 7, 11, 15, 19, \cdots\}$
  Solution: $a_n = 4n - 1$.
  (d) $\left\{\frac{7}{9}, \frac{7}{27}, \frac{7}{81}, \frac{7}{243}, \frac{7}{729}, \frac{7}{3187}, \cdots\right\}$
  Solution: $a_n = \frac{7}{3^{n+1}}$.
  (e) $\left\{\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{5}{32}, \frac{3}{32}, \frac{7}{128}, \frac{1}{32}, \frac{9}{512}, \frac{5}{512} \cdots\right\} = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \frac{7}{128}, \frac{8}{256}, \frac{9}{512}, \frac{10}{1024}, \cdots\right\}$
  Solution: $a_n = \frac{n}{2^n}$.
  (f) $\{1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots\}$
  Solution: $a_n = (-1)^{n-1}$.
  (g) $\left\{0, \frac{3}{8}, \frac{2}{27}, \frac{5}{64}, \frac{4}{125}, \frac{7}{216}, \frac{6}{343}, \frac{9}{512}, \frac{8}{729}, \frac{11}{1000}, \cdots\right\}$
  Solution: $a_n = \frac{n+(-1)^n}{n^3}$

2. Skill 2: limits of sequences

(2) Determine if the sequences is convergent of divergent. If convergent, evaluate the limit.
  (a) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$
  Solution: $\lim_{n\to\infty} \frac{1}{n} = \lim_{x\to\infty} \frac{1}{x} = \boxed{0}$.
  (b) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$
  Solution: $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = \boxed{1}$.
  (c) $\{\sin(n)\}_{n=5}^{\infty}$
  Solution: The function oscillates and the sequence is divergent.
  (d) $\left\{\sin(\frac{1}{n})\right\}_{n=1}^{\infty}$
  Solution: Since the sine function is continouous we have $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin 0 = \boxed{0}$.

(3) Further problems
  (a) Does $\lim_{n\to\infty} \frac{n}{\sqrt{n+1000}}$ exist?
  Solution: No. We have $\frac{n}{\sqrt{n+1000}} = \frac{\sqrt{n}\sqrt{n}}{\sqrt{n+1000}} = \frac{\sqrt{n}}{\sqrt{1+\frac{1000}{n}}} \xrightarrow[n\to\infty]{} \infty$.
  Solution: No. $\frac{n}{\sqrt{n+1000}} = \frac{1}{\sqrt{\frac{n}{n^2}+\frac{1000}{n^2}}} = \frac{1}{\sqrt{\frac{1}{n}+\frac{1000}{n^2}}}$. We can see that the denominator tends to 0 so the sequence diverges to ∞.
  (b) $\lim_{n\to\infty} \frac{n}{2^n} =$
  Solution: We have $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{(\log 2)2^x} = 0$ by l'Hopital.
  (c) (Math 103 final, 2014) Consider the sequence $\{a_n\}_{n=1}^{\infty} = \left\{1, 0, \frac{1}{2}, 0, 0, \frac{1}{3}, 0, 0, 0, \frac{1}{4}, 0, 0, 0, 0, \frac{1}{5}, \cdots\right\}$. Decide whether $\lim_{n\to\infty} a_n = 0$.
  Solution: Yes, the limit is zero.

Date: 3/3/2017, Worksheet by Lior Silberman.

3. Tool: Squeeze Theorem

(4) Determine if each sequence is convergent or divergent. If convergent, evaluate the limit.

(a) (Final 2013) $\left\{(-1)^n \sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$.

Solution: For $n \geq 1$, $\sin\left(\frac{1}{n}\right) \geq 0$ so

$$-\sin\left(\frac{1}{n}\right) \leq (-1)^n \sin\left(\frac{1}{n}\right) \leq \sin\left(\frac{1}{n}\right) .$$

We have seen in 1(d) that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$ and it follows that $\lim_{n\to\infty}\left(-\sin\left(\frac{1}{n}\right)\right) = -\lim_{n\to\infty}\sin\left(\frac{1}{n}\right) = 0$ as well. By the squeeze theorem we conclude that

$$\lim_{n\to\infty}(-1)^n \sin\left(\frac{1}{n}\right) = 0 .$$

(b) (Final 2011) $\left\{\frac{\sin(n)}{\log(n)}\right\}_{n=2}^{\infty}$ (why do we have $n \geq 2$ here?)

Solution: Since $\lim_{n\to\infty}\log(n) = \lim_{x\to\infty}\log(x) = \infty$, we have $\lim_{n\to\infty}\frac{1}{\log n} = 0$. Also, for every n we have $-1 \leq \sin n \leq 1$ so that

$$-\frac{1}{\log n} \leq \frac{\sin n}{\log n} \leq \frac{1}{\log n} .$$

Since $\lim_{n\to\infty} -\frac{1}{\log n} = -\lim_{n\to\infty}\frac{1}{\log n} = 0$ also, we have by the squeeze theorem that

$$\lim_{n\to\infty}\frac{\sin n}{\log n} = 0 .$$

(c) (Math 105 Final 2012) $a_n = 1 + \frac{n!\sin(n^3)}{(n+1)!}$.

Solution: We have $(n+1)! = n!(n+1)$ so $a_n = 1 + \frac{\sin(n^3)}{n+1}$, and for every n we have $-1 \leq \sin(n^3) \leq 1$ so that

$$1 - \frac{1}{n+1} \leq 1 + \frac{\sin(n^3)}{n+1} \leq 1 + \frac{1}{n+1} .$$

Now $\lim_{n\to\infty}\left(1 \pm \frac{1}{n+1}\right) = 1 \pm \lim_{x\to\infty}\frac{1}{x} = 1$ and it follows from the squeeze theorem that

$$\lim_{n\to\infty} 1 + \frac{n!\sin(n^3)}{(n+1)!} = 1 .$$