Math 101 – SOLUTIONS TO WORKSHEET 11
TRIGONOMETRIC INTEGRALS

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(1) Evaluate the integrals

(a) $\int \sin^4 x \cos^3 x \, \mathrm{d}x$

Solution: The power of cosine is odd, while the power of sine is even, so let $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$. Then $\sin^4 x = u^4$, $\cos^2 x = 1 - u^2$ and

$$\begin{aligned}\int \sin^4 x \cos^3 x \, \mathrm{d}x &= \int u^4(1-u^2)\,\mathrm{d}u = \int \left(u^4 - u^6\right)\mathrm{d}u = \frac{u^5}{5} + \frac{u^7}{7} + C \\ &= \frac{1}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C\,.\end{aligned}$$

(b) $\int \sin^5 x \cos^4 x \, \mathrm{d}x$

Solution: This time let $u = \cos x$, $\mathrm{d}u = -\sin x$. Then

$$\begin{aligned}\int \sin^5 x \cos^4 x \, \mathrm{d}x &= \int \left(\sin^2 x\right)^2 \cos^4 x \sin x \, \mathrm{d}x \\ &= -\int \left(1-u^2\right)^2 u^4 \, \mathrm{d}u = \int \left(1 - 2u^2 + u^4\right) u^4 \, \mathrm{d}u \\ &= \int \left(2u^6 - u^4 - u^8\right)\mathrm{d}u = \frac{2}{7}u^7 - \frac{1}{5}u^5 - \frac{1}{9}u^9 + C \\ &= \frac{2}{7}\cos^7 x - \frac{1}{5}\cos^5 x - \frac{1}{9}\cos^9 x + C\end{aligned}$$

(c) $\int \sin^4 x \cos^4 x \, \mathrm{d}x$

Solution: We have $\sin x \cos x = \frac{1}{2}\sin(2x)$ and $\sin^2(2x) = \frac{1-\cos(4x)}{2}$ so

$$\begin{aligned}\int \sin^4 x \cos^4 x \, \mathrm{d}x &= \frac{1}{16}\int \sin^4(2x)\,\mathrm{d}x \\ &= \frac{1}{16}\int \left(\frac{1-\cos(4x)}{2}\right)^2 \mathrm{d}x \\ &= \frac{1}{64}\int \left(1 - 2\cos(4x) + \cos^2(4x)\right)\mathrm{d}x \\ &= \frac{1}{64}\left(x - \frac{2}{4}\sin(4x)\right) + \frac{1}{64}\int \frac{1+\cos(8x)}{2}\,\mathrm{d}x \\ &= \frac{1}{64}x - \frac{1}{128}\sin(4x) + \frac{1}{128}x + \frac{1}{64\cdot 16}\sin(8x) + C \\ &= \frac{3}{128}x - \frac{1}{128}\sin(4x) + \frac{1}{1024}\sin(8x) + C\,.\end{aligned}$$

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Solution: Use $\sin^2 x = \frac{1-\cos(2x)}{2}$, $\cos^2 x = \frac{1+\cos(2x)}{2}$ to get

$$\begin{aligned}
\int \sin^4 x \cos^4 x \,\mathrm{d}x &= \frac{1}{16}\int (1-\cos(2x))^2\,(1+\cos(2x))^2 \\
&= \frac{1}{16}\int ((1-\cos(2x))(1+\cos(2x)))^2\,\mathrm{d}x \\
&= \frac{1}{16}\int \left(1-\cos^2(2x)\right)^2 \mathrm{d}x \\
&= \frac{1}{16}\int \left(1-\frac{1+\cos(4x)}{2}\right)^2 \mathrm{d}x \\
&= \frac{1}{16}\int \left(\frac{1-\cos(4x)}{2}\right)^2 \mathrm{d}x \\
&= \frac{1}{64}\int \left(1-2\cos(4x)+\cos^2(4x)\right)\mathrm{d}x \\
&= \frac{1}{64}\int \left(1-2\cos(4x)+\frac{1+\cos(8x)}{2}\right)\mathrm{d}x \\
&= \frac{3}{128}x - \frac{1}{128}\sin(4x) + \frac{1}{1024}\sin(8x) + C\,.
\end{aligned}$$

(d) $\int \sin^5 x \cos^3 x \,\mathrm{d}x$

Solution: The power of $\cos x$ is odd, so use $u = \sin x$ to get

$$\begin{aligned}
\int \sin^5 x \cos^3 x \,\mathrm{d}x &= \int \sin^5 x \cos^2 x \cos x \,\mathrm{d}x \\
&= \int u^5(1-u^2)\,\mathrm{d}u = \int \left(u^5 - u^7\right)\mathrm{d}u \\
&= \frac{1}{6}u^6 - \frac{1}{8}u^8 + C \\
&= \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C\,.
\end{aligned}$$

Solution: The power of $\sin x$ is odd, so we can also use $u = \cos x$ to get

$$\begin{aligned}
\int \sin^5 x \cos^3 x \,\mathrm{d}x &= \int \left(\sin^2 x\right)^2 \cos^3 x \sin x \,\mathrm{d}x \\
&= -\int (1-u^2)^2 u^3 \,\mathrm{d}u \\
&= \int \left(2u^2 - 1 - u^4\right) u^3 \,\mathrm{d}u \\
&= \int \left(2u^5 - u^3 - u^7\right)\mathrm{d}u \\
&= \frac{2}{6}u^6 - \frac{1}{4}u^4 - \frac{1}{8}u^8 + C \\
&= \frac{1}{3}\cos^6 x - \frac{1}{4}\cos^4 x - \frac{1}{8}\cos^8 x + C\,.
\end{aligned}$$

(2) Powers of tangent and secant

(a) Evaluate $\int_0^{\pi/4} \tan x \,\mathrm{d}x$

Solution: $\int_0^{\pi/4} \tan x = \int_0^{\pi/4} \frac{\sin x}{\cos x}\,\mathrm{d}x$. We note that $\sin x\,\mathrm{d}x$ is roughly $d(\cos x) = -\sin x\,\mathrm{d}x$ so letting $u = \cos x$ we have

$$\begin{aligned}\int_0^{\pi/4} \tan x &= \int_{x=0}^{x=\pi/4} \frac{-\,\mathrm{d}u}{u} = -\left[\log|u|\right]_{u=1}^{u=\cos(\pi/4)} \\ &= -\left(\log(\frac{1}{\sqrt{2}}) - \log 1\right) = -\log\frac{1}{\sqrt{2}} = \boxed{\frac{1}{2}\log 2}.\end{aligned}$$

(b) Evaluate $\int_{-\pi/4}^{+\pi/4} \tan x\,\mathrm{d}x$
Solution: Since $\tan(-x) = \tan(x)$ and the domain is symmetric about $x = 0$, the integral vanishes. In detail, let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$ abd

$$\begin{aligned}\int_{x=-\pi/4}^{x=+\pi/4} \tan x\,\mathrm{d}x &= \int_{u=\pi/4}^{u=-\pi/4} \tan(-u)(-\,\mathrm{d}u) \\ &= \int_{u=\pi/4}^{u=-\pi/4} \tan(u)\,\mathrm{d}u && \tan(-u) = -\tan(u) \\ &= -\int_{u=-\pi/4}^{u=\pi/4} \tan(u)\,\mathrm{d}u && -\int_b^a = \int_a^b \\ &= -\int_{x=-\pi/4}^{x=+\pi/4} \tan(x)\,\mathrm{d}x && \text{substitute } u = x.\end{aligned}$$

(c) (even power of secant) Evaluate $\int \tan^5 x \sec^4 x\,\mathrm{d}x$ using the substitution $u = \tan x$.
Solution: We have $\mathrm{d}u = \sec^2\mathrm{d}x$ so

$$\begin{aligned}\int \tan^5 x \sec^4 x\,\mathrm{d}x &= \int \tan^5 x \sec^2 x \left(\sec^2 x\,\mathrm{d}x\right) \\ &= \int u^5\left(1+u^2\right)\mathrm{d}u = \int \left(u^5 + u^7\right)\mathrm{d}u \\ &= \frac{1}{6}u^6 + \frac{1}{8}u^8 + C = \boxed{\frac{1}{6}\tan^6 x + \frac{1}{8}\tan^8 x + C\,.}\end{aligned}$$

(d) (odd power of tangent) Write $\int \tan^5 x \sec^3 x\,\mathrm{d}x$ in the form $\int \sin^n x \cos^m x\,\mathrm{d}x$ and evaluate it.
Solution: $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$ so we have $\int \frac{\sin^5 x}{\cos^8 x}\,\mathrm{d}x$. Now the power of sine is odd, so we can set $u = \cos x\,\mathrm{d}x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$ and get

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^8 x}\,\mathrm{d}x &= -\int \frac{\sin^4 x}{\cos^8 x}\left(-\sin x\,\mathrm{d}x\right) \\ &= -\int \frac{\left(1-u^2\right)^2}{u^8}\,\mathrm{d}u = -\int \frac{u^4 - 2u^2 + 1}{u^8}\,\mathrm{d}u \\ &= \int \left(-u^{-4} + 2u^{-6} - u^{-8}\right)\mathrm{d}u = \frac{1}{3}u^{-3} - \frac{2}{5}u^{-5} + \frac{1}{7}u^{-7} + C \\ &= \boxed{\frac{1}{3}\cos^{-3} x - \frac{2}{5}\cos^{-5} x + \frac{1}{7}\cos^{-7} x + C\,.} \\ &= \boxed{\frac{1}{3}\sec^3 x - \frac{2}{5}\sec^5 x + \frac{1}{7}\sec^7 x + C\,.}\end{aligned}$$

Solution: The textbook suggests instead substituting $u = \sec x$, where $\mathrm{d}u = \sec x \tan x$ so (also using $\tan^2 x = \sec^2 x - 1$)

$$\begin{aligned}
\int \tan^5 x \sec^3 x \,\mathrm{d}x &= \int \left(\tan^2 x\right)^2 \sec^2 x \,(\tan x \sec x \,\mathrm{d}x) \\
&= \int \left(u^2 - 1\right)^2 u^2 \,\mathrm{d}u = \int \left(u^4 - 2u^2 + 1\right) u^2 \,\mathrm{d}u \\
&= \int \left(u^6 - 2u^4 + u^2\right) \mathrm{d}u = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{u^3}{3} + C \\
&= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 u + C
\end{aligned}$$