30. Power series (22/3/2017)

Goals:

(1) Notion of power series, including identifying centre of expansion.
(2) Interval of convergence.
  (a) Using ratio test to find it.
  (b) Convergence at the edge.

Last time: Ratio test: let $q = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. If $q>1$, $\sum_{n=1}^{\infty} a_n$ diverges, if $q<1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely. [If $q=1$, test is inconclusive).

Challenge: Apply this to $\sum_{n=1}^{\infty} \frac{n!}{2^{n(n-1)}}$.

Ultimate goal: Express functions as sums of series.

Today: Study the series for that.

Examples: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, fact: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

both series involve powers of x, call them "power series".

Def: The power series with centre c (expansion), coefficient A_n, is the series

$$\sum_{n=0}^{\infty} A_n (x-c)^n$$

(c and $\{A_n\}_{n=0}^{\infty}$ are numbers)

In $\sum_{n=0}^{\infty} x^n$, $c=0$, $A_n = 1$

In $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $c=0$, $A_n = \frac{1}{n!}$

Can also write

$$A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \ldots$$

$(x-c)^0 = 1$ for all x, including $0^0 = 1$

Math 101 – WORKSHEET 30
POWER SERIES

(1) Which of the following is a power series:

$\boxed{\checkmark}\displaystyle\sum_{n=0}^{\infty}\frac{n!(x-3)^n}{2^{2n}}$ $\quad\square\displaystyle\sum_{n=0}^{\infty}\frac{3}{n!}(e^x)^n$ ← no - $(e^x)^n$ not power of x.

$C=3$ →
$A_n=\frac{n!}{2^{2n}}$.

$\boxed{\text{Aside: } \sum_{n=0}^{\infty}(2x-1)^n=\sum_{n=0}^{\infty}2^n\left(x-\tfrac{1}{2}\right)^n \text{ is a power serie}}$

1. The interval of convergence

(2) Find the interval of convergence and radius of convergence of the power series

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$; $\left|\frac{(-1)^n\cdot(x-1)^{n+1}}{n+1}\Big/\frac{(-1)^{n-1}\cdot(x-1)^n}{n}\right|=\left|\frac{(x-1)^{n+1}}{(x-1)^n}\cdot\frac{n}{n+1}\right|$

$=|x-1|\cdot\frac{1}{1+\frac{1}{n}}\xrightarrow[n\to\infty]{}|x-1|$

so by ratio test, the series converges for $|x-1|<1$, diverges if $|x-1|>1$

(b) $\sum_{n=0}^{\infty}n!x^n$; $\left|\frac{x^{n+1}(n+1)!}{x^n\cdot n!}\right|=|x|(n+1)\xrightarrow[n\to\infty]{}\infty$ if $x\neq 0$

so the series diverges if $x\neq 0$ (ratio test), interval of convergence is $\{0\}$
$R=0$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$; $\left|\frac{x^{n+1}}{(n+1)!}\Big/\frac{x^n}{n!}\right|=|x|\cdot\left|\frac{n!}{(n+1)!}\right|=\frac{|x|}{n+1}\xrightarrow[n\to\infty]{}0$

So by ratio test, the series converges everywhere (for all x)

$R=\infty$

Date: 22/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Continuation of 2(a): We know the series converges in $(0,2)$, diverges if $|x-1|>1 \Leftrightarrow (-\infty,0)\cup(2,\infty)$

What about endpoints $0, 2$?

At $x=2$ have the series $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}$ which converges by alternating series test (missing details)

At $x=0$, we have the series $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}\cdot(-1)}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges (p-series, $p=1$)

So the interval of convergence is $(0,2]$.

Summary: ratio test $\Rightarrow$ ~~sy~~ open interval, symmetric about centre

need other tests for endpoints

The radius of convergence is the number R s.t. the open interval is $(c-R, c+R)$

~~The interv~~ Caveats: can have $R=0$ (only converge at c)

(1) can have $R=\infty$ (converge everywhere)

(2) if $0<R<\infty$, can have any interval of that radius:

$(c-R, c+R)$ (no convergence at endpoints)

$[c-R, c+R)$, $(c-R, c+R]$ (only one endpoint)

$[c-R, c+R]$ (both endpoints included)

(see last 3 ~~exp~~ examples of worksheet 29)

$$\left|\frac{(x-2)^{n+1}}{3^{n+1}((n+1)^2+1)} \Big/ \frac{(x-2)^n}{3^n(n^2+1)}\right| = |x-2|\cdot\frac{3^n}{3^{n+1}}\cdot\frac{n^2+1}{n^2+2n+2} = \frac{|x-2|}{3}\cdot\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} \xrightarrow[n\to\infty]{} \frac{|x-2|}{3}$$

(d) (Final, 2014, variant) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{3^n(n^2+1)}$

so the series converges if $\frac{|x-2|}{3}<1$, i.e. if $|x-2|<3$, $-1<x<5$, diverges if $|x-2|>3$, need to check $x=-1$, $x=5$. $R=3$

skip check, answer $[-1,5]$

(e) (Final, 2011) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{\log(n+2)}$

(3) Consider a power series $\sum_{n=0}^{\infty} A_n (x-5)^n$.

(a) The power series converges at $x=-3$. Show that it converges at $x=10$.

The centre here is $C=5$. If the series converges at $x=-3$, then its radius R is ≥ 8 so the series converges at least on $(-3,13)$, including at $x=10$

(b) The power series diverges at $x=15$. Show that it diverges at $x=-15$.

Know $R\leq 10$, and $|-15-5|=20>10$ (converge at most on ~~[illegible]~~ $[-5,15]$, not including -15.

(c) Can you tell if the series converges at $x=14$? What can you say about the radius of convergence?

<u>Observation:</u> In $\sum_{n=0}^{\infty} A_n (x-c)^n$, if $L = \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|$ exists, then $R = \frac{1}{L}$. ($\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$ here only)

<u>Fact:</u> Even if ratio test fails, power series <u>always</u> converge in a symmetric interval (+ or - endpoints)

← know this fact.