24. SERIES (7/3/2017)

Goals:

(1) Series: definition, convergence and divergence.
(2) Nth element test.
(3) Integral test.

Last time: ① Geometric series $\left(\sum_{n=0}^{\infty} aq^n\right)$ converge when $|q|<1$,
have $\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}$

② Telescoping series, eg. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right)$
$+ \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{5}}\right) + \ldots = 1$

Both cases, series could be summed exactly.

Today: ~~(1) Review~~ (0) review
(1) series in general, turn to question of convergence.
(2) divergence / N^{th} element test.
(3) integral test - relating series $\leftrightarrow$ improper integrals

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Worksheet ①

Series: An expression $\sum_{n=1}^{\infty} a_n$. Gets meaning by considering partial sums $S_N = a_1 + \ldots + a_N = \sum_{n=1}^{N} a_n$.

Say $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{N\to\infty} S_N$ exists.
diverges if not

If $\sum_{n=1}^{\infty} a_n$ converges write $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$.

Math 101 – WORKSHEET 24
SERIES

1. Review: Geometric and telescoping series

(1) Decide whether the following series converge or diverge

(a) $\sum_{n=5}^{\infty} \frac{\pi^{2n+3}}{9^{n-2}}$; geometric series with ratio $\frac{\pi^2}{9} = \left(\frac{\pi}{3}\right)^2 > 1$
so series diverges

(b) $\sum_{n=5}^{\infty} \frac{e^{2n+2}}{9^{n-2}}$ – ratio $\frac{e^2}{9} = \left(\frac{e}{3}\right)^2 < 1$, series converges

(c) $\sum_{n=1}^{\infty} \left(n^2 - (n+1)^2\right)$ diverges:

$$\sum_{n=1}^{N} \left(n^2-(n+1)^2\right) = (1^2-2^2)+(2^2-3^2)+(3^2-4^2)+\cdots(N^2-(N+1)^2)$$
$$= 1^2-(N+1)^2 \xrightarrow[N\to\infty]{} -\infty$$, so series diverges

(Also, $\sum_{n=1}^{\infty}\left(n^2-(n+1)^2\right) = \sum_{n=1}^{\infty}(-2n-1) = -\sum_{n=1}^{\infty}(2n+1) = -\infty$

Remark: convergence of series is about $\lim_{N\to\infty} S_N$.
not about $\lim_{n\to\infty} a_n$.

Eg $a_n = 1$ for all n $\lim_{n\to\infty} a_n = 1$ exists but $S_N =$

$$S_N = \underbrace{1+1+\cdot+1}_{N} = N \underset{N\to\infty}{\to} \infty, \text{ i.e. } \sum_{n=1}^{\infty} 1 \text{ diverges}$$

But: One-way test: suppose $\sum_{n=1}^{\infty} a_N$ converges:

$\lim_{N\to\infty} S_N = S'$

Then $\lim_{N\to\infty} a_N = \lim_{N\to\infty} (S_N - S_{N-1}) = \left(\lim_{N\to\infty} S_N\right) - \left(\lim_{N\to\infty} S_{N-1}\right) = S' - S' = 0$

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$
↑ terms

$$\boxed{\text{So if } \lim_{n\to\infty} a_n \text{ DNE or is } \neq 0 \text{ then series } \sum_{n=1}^{\infty} a_n \text{ diverges}}$$

false: if $\lim_{n\to\infty} a_n = 0$ then series converges

Message: If terms go to zero, need to think (eg. how fast, ..)
If not, nothing more to think about.

Also: changing / removing / adding finitely many terms does not change fact of convergence / divergence. For a sequence doesn't change limit. For a series, changes the sum

2. Skill 1: elements of a convergent series

(2) Show the following series diverge

(a) $\sum_{n=1}^{\infty}(-1)^n$ $= -1+1-1+1-1+1\ldots$

partial sums: $-1, 0, -1, 0, -1, 0, -1, 0, \ldots \Rightarrow$ no limit

or:

$\{(-1)^n\}_{n=1}^{\infty}$ does not converge ~~to 0~~, so $\sum_{n=1}^{\infty}(-1)^n$ diverges

(b) $\sum_{n=0}^{\infty} n^2 \sin(n)$

For n large, n can be close to any angle so can make $\sin(n)$ close to $1/2$, then $n^2\sin(n)$ large

(hard question)

(c) $\sum_{n=1}^{\infty} \frac{n+\sin n}{n}$

need squeeze thm

$\lim_{n\to\infty} \frac{\sin n + n}{n} = \lim_{n\to\infty}\left(1+\frac{\sin n}{n}\right) = 1 + \lim_{n\to\infty}\frac{\sin n}{n} = 1+0 = 1 \neq 0$

so by nth element test series diverges

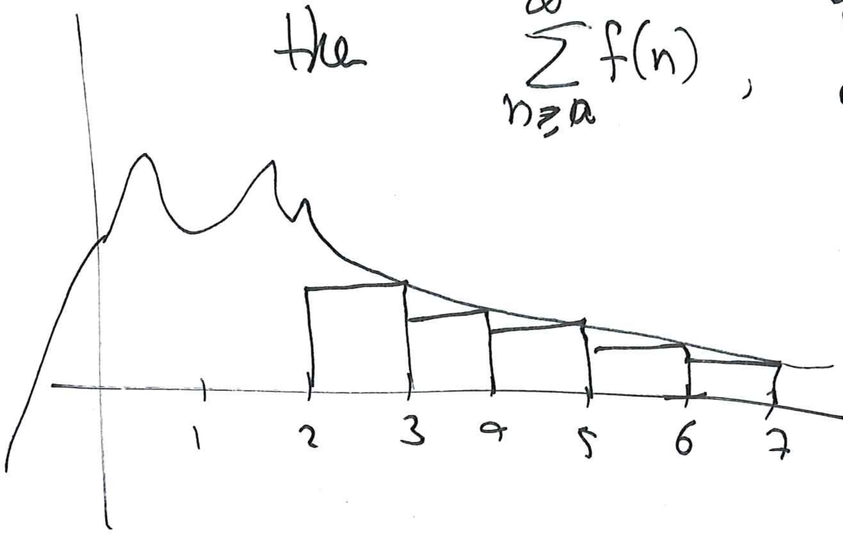
Fact: Suppose $f(x)$ is defined, cts, on $[a, \infty)$.

Suppose $f(x)$ eventually positive, eventually decreasing.

the $\sum_{n \geq a}^{\infty} f(n)$, $\int_a^{\infty} f(x)\,dx$ converge or diverge

together

("integral test")



4. Skill 2: The Integral test

(6) Decide whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (your answer will depend on p).

1(a) let $f(x) = \frac{1}{x}$, then f is positive & decreasing on $[1, \infty)$

~~B~~ $\int_1^{\infty} \frac{dx}{x}$ diverges (p-test) so by integral test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

"Harmonic series"

(b) Want $f(x) = \frac{1}{x^p}$ decreasing (if $p > 0$)

then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if ~~p~~ $p > 1$

diverges if ~~p~~ $p \le 1$

~~If~~ $p \le 0$ then $\frac{1}{n^p}$ does not go to zero