10. Integration by parts continued (25/1/2017)

Goals.

(1) Integration by parts with boundary
(2) Working with a toolkit.

Last time: $\boxed{\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x}$ $\frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C$

Problem: Evaluate the integrals

$\int x\log x\,\mathrm{d}x = \frac{1}{2}x^2\log x - \int \frac{1}{x}\cdot\frac{1}{2}x^2 dx = \frac{1}{2}x^2\log x - \int \frac{1}{2}x\,dx =$

try by parts, $u=x$, $dv = \log x\,dx$ need anti-derivative of $\log x$.

Problem - try something else: $u = \log x$, $dv = x\,dx$

$du = \frac{1}{x}$ $v = \frac{1}{2}x^2$

$\int_1^2 x\log x\,\mathrm{d}x =$

Method 1: compute anti-derivative (here $\frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C$)

get answer:

$$\int_1^2 x\log x\,dx = \left[\frac{1}{2}x^2\log x - \frac{1}{4}x^2\right]_1^2 =$$

$$=(2\log 2 - 1) - \left(-\frac{1}{4}\right) = 2\log 2 - \frac{3}{4}.$$

Method 2: Integrate by parts

$$\int_1^2 x\log x\,dx = \left[\frac{1}{2}x^2\log x\right]_{x=1}^{x=2} - \int_1^2 \frac{1}{2}x\,dx$$

$$=2\log 2 - 0 - \left[\frac{1}{4}x^2\right]_1^2 = 2\log 2 - \frac{3}{4}$$

Math 101 – WORKSHEET 10
INTEGRATION BY PARTS

(1) Evaluate the following integals:

(a) $\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx =$

by parts

$u = x^2, dV = \cos x \, dx$

$du = 2x \, dx, \quad V = \sin x$

by parts

$= x^2 \sin x - 2x(-\cos x) + 2 \cdot \int (-\cos x) \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$

$u = x, \ dv = \sin x \, dx$

$du = dx, \ V = -\cos x$

persistence: keep working until done

(b) $\int \log x \, dx = x \log x - \int x \cdot \frac{1}{x} \, dx = x \log x - \int 1 \, dx = x \log x - x + C$

by parts

$u = \log x \quad dv = 1 dx$

$du = \frac{1}{x} dx \quad v = x$

Date: 25/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) (Final, 2013) $\int_0^1 \arctan x \, dx = \left[x \arctan x\right]_0^1 - \int_0^1 \frac{x\,dx}{1+x^2} =$

by parts,

$u = \arctan x \quad dv = 1 \cdot dx$

$du = \frac{1}{1+x^2} \quad v = x$

substitution $u = 1+x^2$

$du = 2x$

$$= (1 \cdot \arctan 1 - 0) - \int_{u=1}^{u=2} \frac{\frac{1}{2}du}{u} = \arctan 1 - \frac{1}{2}\left[\log u\right]_{u=1}^{u=2}$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\log 2 - \log 1\right] = \frac{\pi}{4} - \frac{1}{2}\log 2$$

Can also find anti-derivative:

$$\int \arctan x = x\arctan x - \int \frac{x\,dx}{1+x^2} = x\arctan x - \int \frac{\frac{1}{2}du}{u} =$$

$$= x\arctan x - \frac{1}{2}\log u + C = x\arctan x - \frac{1}{2}\log(1+x^2) + C$$

Check: $\frac{d}{dx}\left(x\arctan x - \frac{1}{2}\log(1+x^2)\right) = \arctan x + \cancel{\frac{x}{1+x^2}} - \cancel{\frac{2x}{2(1+x^2)}} = \arctan x.$

(2) Now let's play with our toolkit

(a) Evaluate $\int \frac{\log x}{x}\,\mathrm{d}x$ $= (\log x)^2 - \int \frac{\log x}{x}\,dx$

Try by parts: $u = \log x$, $dv = \frac{dx}{x}$

$du = \frac{1}{x}dx$, $v = \log x$

Try substitution: $u = \log x$, $du = \frac{dx}{x}$ so $\int \frac{\log x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C$

$= \frac{1}{2}(\log x)^2 + C$

(b) (Quiz 2015) Evaluate $\int \frac{\log x}{x^3}\,\mathrm{d}x$ $\underset{\substack{\uparrow \\ \text{by parts}}}{=} -\frac{\log x}{2x^2} + \frac{1}{2}\int \frac{dx}{x^3}$

$u = \log x$, $dv = \frac{1}{x^3}$

$du = \frac{1}{x}dx$, $v = -\frac{1}{2x^2}$

$= -\frac{\log x}{2x^2} - \frac{1}{4x^2} + C$

(c) (Final, 2010) Let $g(x) = \int_0^1 (xe^t - t)^2\,\mathrm{d}t$. Find the minimum value of $g(x)$.

Let's find $g(x)$

$$g(x) = \int_0^1 (xe^t - t)^2\,dt = \int_0^1 (x^2e^{2t} - 2xte^t + t^2)\,dt$$

$$= x^2\int_0^1 e^{2t}\,dt - 2x\int_0^1 te^t\,dt + \int_0^1 t^2\,dt$$

$$\int_0^1 e^{2t}\,dt = \left[\tfrac{1}{2}e^{2t}\right]_0^1 = \tfrac{1}{2}(e^2-1), \quad \int_0^1 t^2\,dt = \left[\tfrac{1}{3}t^3\right]_0^1 = \tfrac{1}{3}$$

$$\int_0^1 te^t\,dt \underset{\text{by parts}}{=} \left[te^t\right]_0^1 - \int_0^1 1\cdot e^t\,dt = e - (e-1) = 1, \text{ so } g(x) = \tfrac{1}{2}(e^2-1)x^2 - 2x + \tfrac{1}{3}$$

(d) Evaluate $\int x^3 \log(x^2+1)\,\mathrm{d}x$