7. Area between curves (18/1/2017)

Goals.

(1) Quiz information
(2) Area between curves
  (a) Geometry – draw pictures
  (b) Write integral – chop and sum
  (c) Evaluate integral
  (d) Final answer

Last Time: Integration by substitution

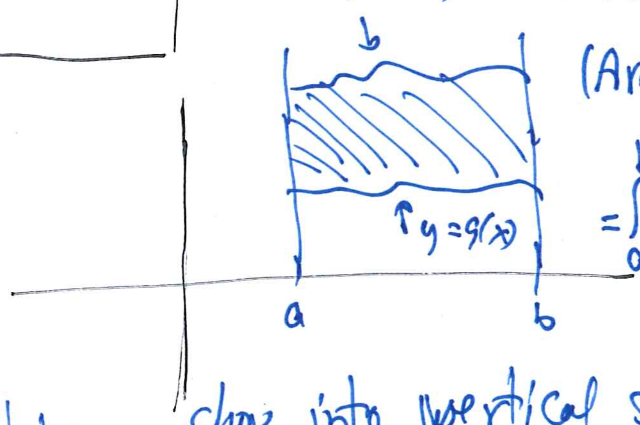
"chain rule in reverse": try to write integrand as $f(g(x))g'(x)$

Today: Find area of plane region between $y=f(x)$, $y=g(x)$

Formula: Area between graphs of $y=f(x)$, $y=g(x)$ and lines $x=a$, $x=b$ (suppose $f(x)\geq g(x)$ on $[a,b]$, $b\geq a$) is:

$$\boxed{\int_a^b (f(x)-g(x))\,dx}$$

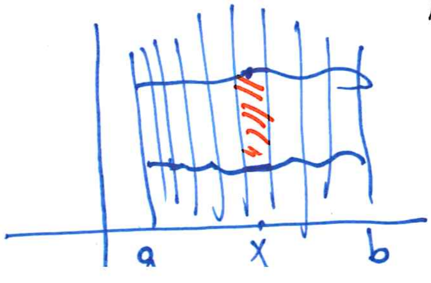
Why? Answer 1:



Area between f,g = (Area under f) − (Area under g)

$$=\int_a^b f(x)dx - \int_a^b g(x)dx$$

Answer 2:

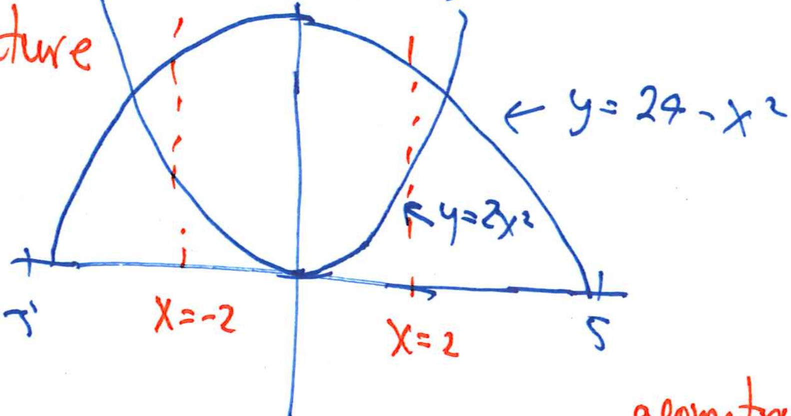


chop into vertical slices
slice at x is approx. rectangle of width dx height $\approx f(x)-g(x)$
area element is $(f(x)-g(x))dx$.

Now integrate from a to b

__Example:__ Find area between graphs of $y = 24 - x^2$, $y = 2x^2$ and the lines $x = 2$, $x = -2$

(1) Draw picture



(intersection pts where $24 - x^2 = 2x^2$
$\Updownarrow$
$3x^2 = 24$
$\Updownarrow$
$x = \pm\sqrt{8} = \pm 2\sqrt{2}$)

$\leftarrow$ geometry

On $[-2, 2]$, $24 - x^2 \geq 2x^2$, so area is

(2) setup integral $\rightarrow$ $$\int_{-2}^{2} \left((24 - x^2) - (2x^2)\right) dx = \int_{-2}^{2} (24 - 3x^2)\, dx = \left[24x - x^3\right]_{-2}^{2} =$$

$$= (48 - 8) - (-48 + 8) = 80$$

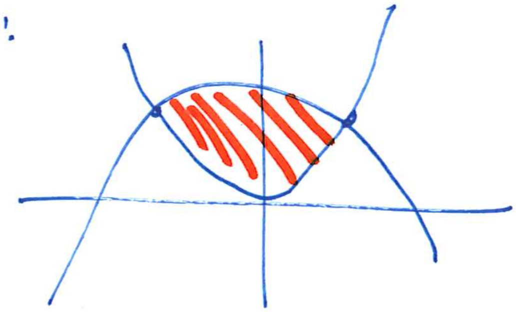
(3) evaluate the integral

__Q:__ What if we don't know order?

__A:__ Yes, area is always $\displaystyle\int_a^b |f(x) - g(x)|\, dx$

What if I ask for "the area between graphs of $\begin{cases} y = 2x^2 \\ y = 24 - x^2 \end{cases}$"?

That would mean:

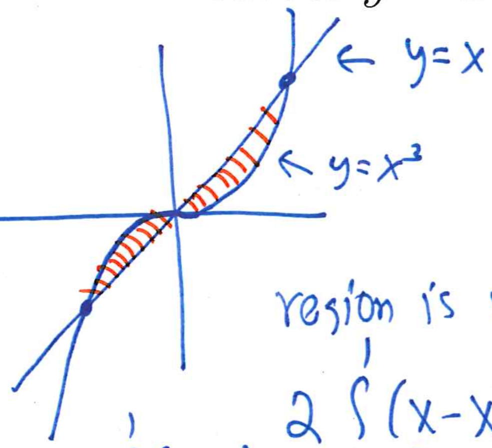


(extra geometric step: find intersection pts)

Math 101 – WORKSHEET 7
AREA BETWEEN CURVES

(1) Find the total area of the following planar regions. It will be useful to sketch the region first.

(a) (Final, 2011) The finite region lying between the curves $y = x$ and $y = x^3$.



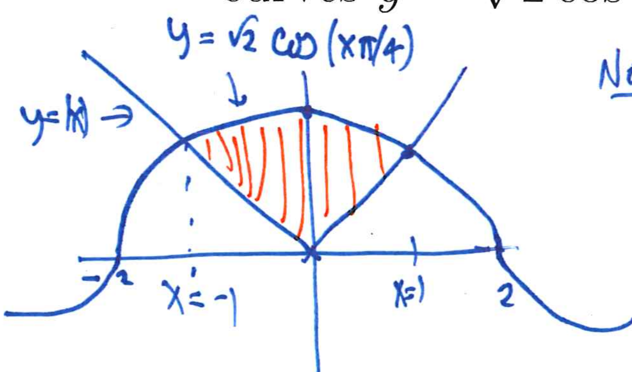
curves intersect where $x = x^3$, i.e. if $x^3 - x = x(x-1)(x+1) = 0$, i.e. at $x = -1, 0, 1$.

region is symmetric under reflection, so area is

$$2\int_0^1 (x - x^3)\,dx = 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}.$$

(Also do $\int_0^1 (x-x^3)\,dx + \int_{-1}^0 (x^3 - x)\,dx$)

(b) (Final, 2014) The finite region bounded by the two curves $y = \sqrt{2}\cos(x\pi/4)$ and $y = |x|$.



Note: at $x = \pm 1$, $\cos\left(\frac{x\pi}{4}\right) = \cos\left(\pm\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

so $\sqrt{2}\cos\left(\frac{x\pi}{4}\right) = 1 = |\pm 1|$

region is symmetric about y-axis, so total area is

$$2\int_0^1 \left(\sqrt{2}\cos\left(\frac{x\pi}{4}\right) - x\right)dx =$$

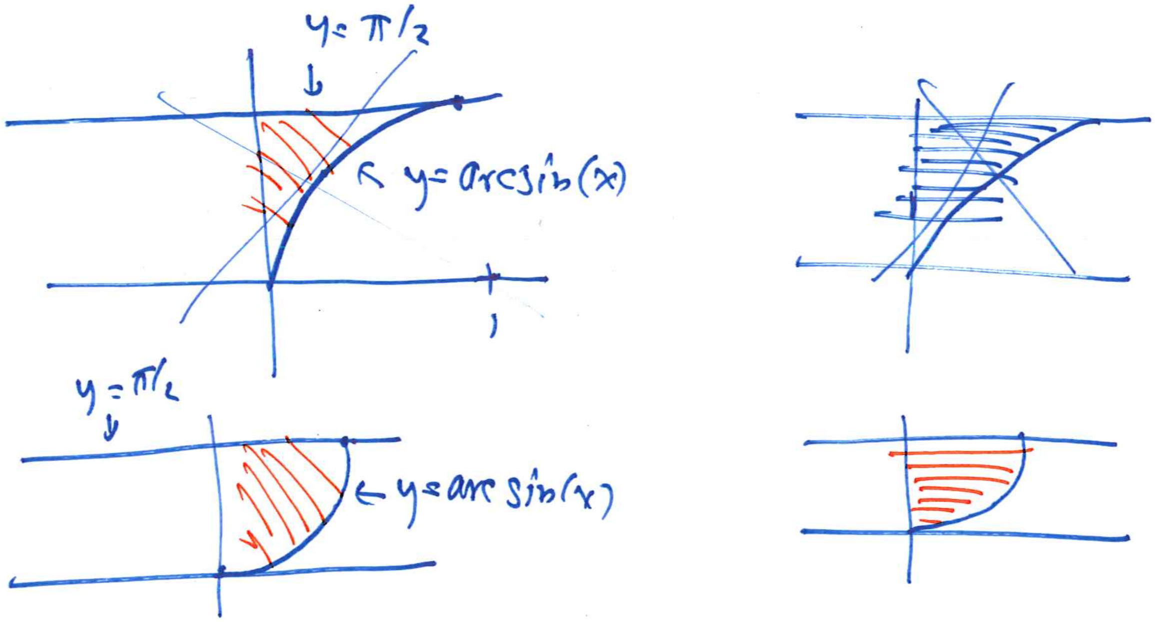
Date: 18/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$$= 2\sqrt{2}\int_0^1 \cos\left(\frac{x\pi}{4}\right)dx - \int_0^1 2x\,dx = 2\sqrt{2}\int_{x=0}^{x=1} \cos u \left(\frac{4}{\pi}du\right) - \left[x^2\right]_0^1$$

$$= \frac{8\sqrt{2}}{\pi}\left[\sin u\right]_{x=0}^{x=1} - \left[x^2\right]_{x=0}^{x=1} = \frac{8\sqrt{2}}{\pi}\sin\left(\frac{\pi}{4}\right) - 1 \quad \left| u = \frac{x\pi}{4},\ du = \frac{\pi}{4}dx \right| = \frac{8}{\pi} - 1$$

(2) Find the total area of the following planar regions. It will be useful to sketch the region first.

(a) The finite region bounded by the y-axis, the graph of $y = \arcsin(x)$ and the line $y = \frac{\pi}{2}$.



(b) (Quiz, 2015) The finite region to the left of the y-axis and to the right of the curve $x = y^2 + y$.