Lior Silberman's Math 539: Problem Set 1 (due 1/2/2014)

The main challenges are problem 3, 4, 8(c) and 9.

NOTATION $f = \Theta(g)$ means $f = O(g)$ and $g = O(f)$, that is $0 \leq \frac{1}{C} f(x) \leq g(x) \leq C f(x)$ for all large enough x.

1. (The standard divisor bound)
   (a) Let $f(n)$ be multiplicative, and suppose that $f \to 0$ along prime powers – that is, for every $\varepsilon > 0$ there is N such that if $p^m > N$ then $|f(p^m)| \leq \varepsilon$. Show that $\lim_{n\to\infty} f(n) = 0$.
   (b) Show that for all $\varepsilon > 0$, $d(n) = O(n^{\varepsilon})$.

2. Establish some of the following identities in the ring of formal Dirichlet series
   (a) Let $d_k(n) = \sum_{\prod_{i=1}^k a_i = n} 1$ be the generalized divisor functions, counting factorizations of n into k parts (so $d_2(n) = d(n)$ is the usual divisor function). Show $\sum_n d_k(n) n^{-s} = (\zeta(s))^k$ and that $d_k * d_l = d_{k+l}$.
   (b) Define $d_{1/2}(n)$. Calculate $d_{1/2}(p)$, $d_{1/2}(12)$.
   (c) Generalize to a multiplicative function $d_z(n)$ such that $d_z * d_w = d_{z+w}$ and evaluate $d_z(p)$, $d_z(p^k)$, $d_z(360)$.
   (d) Let $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ be the generalized sum-of-divisors function. Show that $\sum_{n=1}^{\infty} \sigma_\alpha(n) n^{-s} = \zeta(s)\zeta(s-\alpha)$.
   (e) Show that $\sum_{n\geq 1} d(n^2) n^{-s} = \frac{\zeta(s)^3}{\zeta(2s)}$.

3. (Averaging)
   (a) Show the function $\frac{\phi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right)$ has the average value $\frac{1}{\zeta(2)}$.
   (b) Show that for each $k \geq 1$ $\frac{1}{x}\sum_{n\leq x} d_k(n) = P_k(\log x) + O(x^{-\frac{1}{k}})$ where P_k is a polynomial of degree $k-1$.
   (c) Consider $\frac{1}{x}\sum_{n\leq x}\sigma_\alpha(n)$. (simplest) Show that $\frac{1}{x}\sum_{n\leq x}\sigma_\alpha(n) = \Theta(x^\alpha)$ (better) Find $C = C(\alpha)$ such that $\frac{1}{x}\sum_{n\leq x}\sigma_\alpha(n) = Cx^\alpha + o(x^\alpha)$ (best) Show $\frac{1}{x}\sum_{n\leq x}\sigma_\alpha(n) = Cx^\alpha + O\left(x^\beta\right)$ for some $\beta < \alpha$.

4. Let $D_\phi(s) = \sum_{n\geq 1}\phi(n) n^{-s}$.
   (a) Represent the series in terms of $\zeta(s)$ formally.
   (b) Show the series converges absolutely for $\Re(s) > 2$.
   (c) Show that the series failes to converge at $s = 2$.

5. Recall the function $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$.
   (a) ("Asymptotic expansion") Show that for fixed K, $\mathrm{Li}(x) = \sum_{k=1}^{K}(k-1)!\frac{x}{\log^k x} + O_K\left(\frac{x}{\log^{K+1} x}\right)$.
   (b) (Asymptotic expansions are not series expansions) Show that $\sum_{k=1}^{\infty}(k-1)!\frac{x}{\log^k x}$ diverges.
   (c) Use summation by parts to estimate $\pi(x) = \sum_{p\leq x} 1$ using the known asymptotics for $\sum_{p\leq x}\frac{1}{p}$. Can you show $\pi(x) \ll \mathrm{Li}(x)$? $\pi(x) \gg \mathrm{Li}(x)$? That $\frac{\pi(x)}{\mathrm{Li}(x)} = 1 + o(1)$?
   (d) Deduce $\pi(x) = \Theta(\mathrm{Li}(x))$ from Chebychev's estimate $\theta(x) = \Theta(x)$.

6. (Chebychev's lower bound) Let $\theta(x) = \sum_{p \le x} \log p$. We will find an explicit $c > 0$ such that $\theta(x) \ge cx$ for $x \ge 2$.
   (a) Let $v_p(n)$ denote the number of times p divides n. Show that $v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$.
   (b) Show that if $n < p \le 2n$ then $v_p\left(\binom{2n}{n}\right) = 1$.
   (c) (main saving) Show that if $\frac{2}{3}n < p \le n$ then $v_p\left(\binom{2n}{n}\right) = 0$ unless $n = p = 2$.
   (d) Show that if $\sqrt{2n} < p \le n$ then $v_p\left(\binom{2n}{n}\right) \le 1$.
   (e) Show that $v_p\left(\binom{2n}{n}\right) \le \log_p 2n$.
   (f) Show that $\log\binom{2n}{n} - (\theta(2n) - \theta(n)) \le \theta\left(\frac{2n}{3}\right) + 2\sqrt{2n}\log(2n)$.
   (g) Find a constant $c > 0$ such that $\theta(x) \ge cx$ for $2 \le x \le 4$ and such that if $\theta(x) \ge cx$ for all $2 \le x \le X$ then $\theta(x) \ge cx$ for $X < x \le 2X$.

7. Notation: $f(x) = o(g(x))$ ("little oh") if $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0$.
   (a) Show $\prod_p \left(1 - \frac{1}{p}\right) e^{\frac{1}{p}}$ converges.
   (b) Show that $\prod_{p \le z} \left(1 - \frac{1}{p}\right) = \frac{C(1+o(1))}{\log z}$.

8. Let $\sigma > 0$
   (a) Show that $\prod_{p \le x}(1 + p^{-\sigma}) \le \exp\left(O(x^{1-\sigma}/\log x)\right)$.
   (b) Let $a_p \in \mathbb{C}$ satisfy $\left|a_p\right| \le p^{-\sigma}$. Show that $f(n) = \prod_{p|n}(1 + a_p) \le \exp\left(O\left((\log^{1-\sigma} n)(\log\log n)^{-1}\right)\right)$.
   (c) Show that $\sum_{n \le x} f(n) = cx + O(x^{1-\sigma})$ where $c = \prod_p \left(1 + \frac{a_p}{p}\right)$.

9. Let $\mathcal{A}_n$ denote a set of representative for the isomorphism classes of abelian groups of order n, $A_n = \#\mathcal{A}_n$ the number of isomorphism classes.
   (a) Show that $\sum_{n \ge 1} A_n n^{-s} = \prod_{k=1}^{\infty} \zeta(ks)$ in the ring of formal Dirichlet series.
   (b) Show that $\sum_{n \le x} A_n = cx + O\left(x^{1/2}\right)$ where $c = \prod_{k=2}^{\infty} \zeta(k)$.

Hint for 5(a): Repeatedly integrate by parts, and for the error estimate effectuate the mantra "log is a constant function" by breaking the interval of integration in two.