Math 322: Problem Set 11 (not for submission)

Nilpotent groups

1. Fix a field F (say $F = R$) and $n \geq 2$. Let $U_n \subset \mathrm{GL}_n(F)$ denote the group of upper-triangular matrices with 1s on the main diagonal. Write E^{ij} for the matrix having 1 at position ij and 0 everywhere else.
   (a) Show that $Z(U_n) = U_{n,n-1} = \{I_n + zE^{1,n} \mid z \in F\}$, the matrices whose only non-zero entry (above the main diagonal) is in the upper right corner.
   (b) Show that the equivalence class of $u \in U_n$ in $U_n/Z(U_n)$ depends exactly on the entries of u_n except the corner one.
   (c) Show that $U_{n,n-2} = \{u \in U_n \mid 2 \leq j < i+n-2 \to u_{ij} = 0\} = \{I_n + \sum_{j-i \geq n-2} z_{ij}E^{ij}\}$ is the subgroup $Z^2(U_n) \lhd U_n$ which contains $Z(U_n)$ and such that $Z^2(U_n)/Z(U_n)$ is the center of $U_n/Z(U_n)$.
   (d) For each $1 \leq m \leq n-1$ let

$$U_{n,m} = \{u \in U_n \mid 2 \leq j < i+m \to u_{ij} = 0\} = \left\{ I_n + \sum_{j-i \geq m} z_{ij}E^{ij} \mid z_{ij} \in F \right\}$$

   be the group with non-zero entries starting in the mth diagonal above the main diagonal (note that $U_{n,1} = U_n$). Show that $U_{n,m}$ is normal in U_n.
   (e) Show that $U_{n,m}/U_{n,m+1}$ is the center of $U_n/U_{n,m+1}$ and conclude that $Z^i(U_n) = U_{n,n-i}$ and that U_n is nilpotent.

DEFINITION. For $A, B \subset G$ write $[A,B]$ for the subgroup $\langle \{[a,b] \mid a \in A, b \in B\} \rangle$ generated by all commutators of elements from A, B.

2. (Descending central series) Let $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$.
   (a) Show by induction that $\gamma_i(G)$ are normal subgroups such that $\gamma_{i+1}(G) \subset \gamma_i(G)$.
   (b) Show that $\gamma_i(G)/\gamma_{i+1}(G)$ is contained in the centre of $G/\gamma_{i+1}(G)$.
   (c) Suppose G is nilpotent of degree d, so that $Z^d(G) = G$. Show that $\gamma_i(G) \subset Z^{d+1-i}(G)$.

Solvable groups

3. Let G be a group of order $p^a q^b$. In each case show that G is solvable (hint: find a normal subgroup N and consider N and G/N separately).
   (a) $a = 2, b = 1$.
   (b) $a = 2, b = 2$.

4. Let $n \geq 2$ and let $B_n(F) \subset \mathrm{GL}_n(F)$ be the subgroup of upper-triangular invertible matrices.
   (a) Show that $U_n \lhd B_n$ and that $B_n/U_n \simeq (F^\times)^n$.
   (b) Show that $B_n(F)$ is solvable.
   (c) Show that (unless $F = \mathbb{F}_2$) $Z(B_n(F))$ consists exactly of the scalar matrices with non-zero entries.
   (d) Show that for a large enough field F, $Z(B_n/Z(B_n)) = \{e\}$. Conclude that B_n is solvable but not nilpotent (this holds for any $F \neq \mathbb{F}_2$).

The derived series

5. Fix a group G. The subgroup $G' = [G, G]$ is called the *derived subgroup*.
   (a) Show that G' is a normal subgruop of G.
   (b) Let $N \triangleleft G$. Show that G/N is abelian iff $G' \subset N$.

   DEF The descending series of subgroups defined by $G^{(0)} = G$ and $G^{(i+1)} = \left(G^{(i)}\right)'$ is called the *derived series*.

   (c) Show that $G^{(i)}/G^{(i+1)}$ is abelian.

6. Let $G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_k$ be a descending series of subgroups of G with G_{i-1}/G_i abelian. Note that we don't assume $G_k = \{e\}$.
   (a) Writing $G^{(1)} = G'$ show that $G^{(1)} \subset G_1$.
   (b) Writing $G^{(2)} = (G')'$ show that $G^{(2)} \subset G_1' \subset G_2$.
   (c) Writing $G^{(i+1)} = \left(G^{(i)}\right)'$ show by induction that $G^{(i)} \subset G_i$ for each i.
   (d) Show that G is solvable iff $G^{(n)} = \{e\}$ for some n.