Math 101 – SOLUTIONS TO WORKSHEET 20
THE CENTRE OF MASS

1. Point masses

(1) Three masses are placed at the points $(-1,0),(1,0),(0,5)$. Find the centre of mass of the configuration.

(a) When the masses are equal,

Solution: The configuration is symmetric upon reflection at the y-axis so the centre of mass is on the axis. Its y-coordinate is

$$\frac{1\cdot 0+1\cdot 0+1\cdot 5}{1+1+1}=\frac{5}{3}$$

so the centre of mass is at $\left(0,\frac{5}{3}\right)$.

(b) When the mass at $(-1,0)$ is twice as large as the others.

Solution: The weights are now $2,2,1$ so the centre of mass is at

$$\begin{aligned} X &= \frac{2(-1)+1(1)+1(0)}{4}=-\frac{1}{4} \\ Y &= \frac{2(0)+1(0)+1(5)}{4}=\frac{5}{4}. \end{aligned}$$

Could also average with the relative weights $\frac{2}{4},\frac{1}{4},\frac{1}{4}=\frac{1}{2},\frac{1}{4},\frac{1}{4}$.

(2) The mass of the Earth is about 6×10^{24}kg. The mass of the Moon is about 7.2×10^{22}kg. The distance between the centres of the Earth and the Moon is $3.8\cdot 10^5$km. Where is the centre of mass of the Earth–Moon system? [aside: the radius of the Earth is about 6400km].

Solution: Let's measure distances from the center of the Earth. Then the centre of mass is at distance

$$\begin{aligned} \frac{x_{\text{Earth}}M_{\text{Earth}}+x_{\text{Moon}}M_{\text{Moon}}}{M_{\text{Earth}}+M_{\text{moon}}} &= \frac{M_{\text{Moon}}}{M_{\text{Earth}}+M_{\text{moon}}}x_{\text{Moon}} \\ &\approx \frac{7.2\times 10^{22}}{6\times 10^{24}+7.2\times 10^{22}}3.8\cdot 10^5\text{km} \\ &\approx \frac{7.2}{6}\cdot 10^{-2}\cdot 3.8\cdot 10^5\text{km} \\ &= (1.2\cdot 3.8)\cdot 10^3\text{km}\approx 4.5\cdot 10^3\text{km}. \end{aligned}$$

In particular the centre of mass is *inside the Earth*.

(3) A tenderizing hammer consists of a 1kg head attached to a 30cm-long shaft massing 400g.

(a) Find the centre of mass of the hammer.

Solution: Let the axis run along the shaft, starting at the handle. The CM of the shaft is 15cm in, so the CM of the hammer is at

$$\frac{400\cdot 15+1,000\cdot 30}{1,400}=\frac{2}{7}\cdot 15+\frac{5}{7}\cdot 30=\frac{180}{7}\approx 25.7\text{cm}$$

along the shaft, that is at distance ≈ 4.3cm from the head.

Solution: Let the axis run along the shaft, starting from the head. The CM of the shaft is 15cm from the head, while the head is at the origin, so the CM is at

$$\frac{400\cdot 15+1,000\cdot 0}{1,400}=\frac{2}{7}\cdot 15\approx 4.3\text{cm}$$

along the shaft, that is at distance ≈ 4.3cm from the head.

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(b) What fraction of the mass of the hammer is on each side of the centre of mass?
Solution: The fraction of the shaft on one side of the CM is $\frac{180/7}{30}$, so the mass is $\frac{6}{7} \cdot 400$ which is only $\frac{6}{7} \cdot \frac{400}{1,400} = \frac{12}{49}$ fraction of the total mass (about $\frac{12}{48} = \frac{1}{6}$!).
WARNING: It's not true that there is half the mass on either side!

2. Regions

(3) (Final 2013) The region R consists of a semicircle of radius 3 on top of a rectangle of width 6 and height 2. Find its centre of mass.

(a) Using the formulas above
Solution: The region is symmetric on reflection in the y-axis, so the CM lies on the axis. The region lies between the graphs of $y = \sqrt{9 - x^2}$ and $y = -2$ on $[-3, 3]$. Its area is $\frac{1}{2}\pi \cdot 3^2 + 6 \cdot 2 = 12 + \frac{9}{2}\pi$ so the Y-coordinate of the CM is at

$$\begin{aligned} Y &= \frac{1}{12 + 4.5\pi} \cdot \frac{1}{2} \int_{-3}^{3} \left((9 - x^2) - (-2)^2\right) \mathrm{d}x = \\ &= \frac{1}{24 + 9\pi} \int_{-3}^{3} \left(5 - x^2\right) \mathrm{d}x = \frac{1}{24 + 9\pi} \left[5x - \frac{x^3}{3}\right]_{-3}^{3} \\ &= \frac{1}{24 + 9\pi} \left[15 - \frac{27}{3} - (-15) + \frac{-27}{3}\right] = \frac{12}{24 + 9\pi} = \boxed{\frac{4}{8 + 3\pi}}. \end{aligned}$$

(b) Using the known locations of the centres of mass of the semicircle and the rectangle.
Solution: We saw in lecture that the CM of a semicircle of radius 1 is at $\left(0, \frac{4}{3\pi}\right)$. Rescaling by 3 we see that the CM of the semicircle in the problem (which has radius 3) is at $\left(0, \frac{4}{\pi}\right)$. The CM of the rectangle is at $(0, -1)$. The total area is still $12 + \frac{9}{2}\pi$ and we can now take the weighted average and get for the Y-coordinate (the X-coordinate is still zero):

$$Y = \frac{12}{12 + \frac{9}{2}\pi}(-1) + \frac{\frac{9}{2}\pi \cdot \frac{4}{\pi}}{12 + \frac{9}{2}\pi} = \frac{18 - 12}{12 + \frac{9}{2}\pi} = \frac{12}{24 + 9\pi} = \boxed{\frac{4}{8 + 3\pi}}.$$

(4) Find the centre of mass of the region lying below the x axis, between the branches of $\log |x|$.
Solution: The logarithm cross the x-axis where $\log |x| = 0$, that is where $x = \pm 1$. Using the symmetry, the area of the region is

$$-\int_{-1}^{1} \log |x| \,\mathrm{d}x = -2 \int_{0}^{1} \log x \,\mathrm{d}x$$

(note that the region lies entirely below the axis, so its area is minus the integral). This is an improper integral; we compute it using integration by parts:

$$\begin{aligned} \int \log x \,\mathrm{d}x &= \int 1 \cdot \log x \,\mathrm{d}x = x \log x - \int x \frac{1}{x} \,\mathrm{d}x = x \log x - \int 1 \,\mathrm{d}x \\ &= x \log x - x + C\,. \end{aligned}$$

It follows that

$$\int_{\epsilon}^{1} \log x \,\mathrm{d}x = \left[x \log x - x\right]_{\epsilon}^{1} = (1 \log 1 - 1) - (\epsilon \log \epsilon - \epsilon)\,.$$

Now letting $\epsilon \to 0$ we have $\lim_{\epsilon \to 0} \epsilon = 0$ and by l'Hôpital

$$\lim_{\epsilon \to 0} \epsilon \log \epsilon = \lim_{\epsilon \to 0} \frac{\log \epsilon}{1/\epsilon} = \lim_{\epsilon \to 0} \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon \to 0} (-\epsilon) = 0\,.$$

It follows that

$$\int_{0}^{1} \log x \,\mathrm{d}x = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \log x \,\mathrm{d}x = -1$$

and the area of the region is 2 (recall that it was $2\int_0^1 \log x \, \mathrm{d}x$).

By the symmetry the CM is on the y axis. To find its y-coordinate we evaluate

$$\frac{1}{2}\int_{-1}^{1} \left(0^2 - \log^2|x|\right) \mathrm{d}x = -\frac{1}{2}\int_{-1}^{1} \log^2 x \, \mathrm{d}x = -\int_0^1 \log^2 x \, \mathrm{d}x$$

(again using the symmetry, to convert an integral on $[-1,1]$ to twice the integral on $[0,1]$). This is improper at zero and we integrate by parts:

$$\begin{aligned} \int \log^2 x \, \mathrm{d}x &= x\log^2 x - \int x\left(2\log x \frac{1}{x}\right) \underset{\cdot}{\mathrm{x}} = x\log^2 x - 2\int \log x \, \mathrm{d}x \\ &= x\log^2 x - 2\left(x\log x - x\right) + C\,. \end{aligned}$$

We therefore have (using $\log 1 = 0$) that

$$\begin{aligned} \int_\epsilon^1 \log^2 x \, \mathrm{d}x &= \left[x\log^2 x - 2x\log x + 2x\right]_{x=\epsilon}^{x=1} \\ &= 2 - \epsilon\log^2\epsilon - 2\epsilon\log\epsilon + 2\epsilon\,. \end{aligned}$$

Finally we take the limit as $\epsilon \to 0$. We already know that $\epsilon \to 0$ and that $\epsilon\log\epsilon \to 0$. It remains to evaluate

$$\lim_{\epsilon\to 0} \epsilon\log^2\epsilon = \lim_{\epsilon\to 0} \frac{\log^2\epsilon}{1/\epsilon} = \lim_{\epsilon\to 0} \frac{2\log\epsilon \cdot \frac{1}{\epsilon}}{-1/\epsilon^2} = -2\lim_{\epsilon\to 0} \epsilon\log\epsilon = 0\,.$$

We conlcude that

$$\int_0^1 \log^2 x \, \mathrm{d}x = 2$$

and therefore $-\int_0^1 \log^2 x \, \mathrm{d}x = -2$. To find the CM we need to divide by the area (which was 2) so the CM is at

$$\boxed{(0,-1)\ .}$$