MATH 100 – NOTES 19
THE SHAPE OF THE GRAPH

1. Tools

Let f be differentiable as needed on (a, b).

Fact (First derivative)**.** (1) *If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly* increasing *there.*
(2) *If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly* decreasing *there.*

Every change involves either a *critical point* (f' vanishes) or a *singularity* (f' undefined).

Fact (Second derivative)**.** (1) *If $f''(x) > 0$ for all $x \in (a, b)$ then f is concave* up *there.*
(2) *If $f''(x) < 0$ for all $x \in (a, b)$ then f is concave* down *there.*

Definition 1. A change in concavity is called an *inflection point.*

Theorem. *(Tests for minima and maxima) Let $x_0 \in (a, b)$ be a critical or singular number for f, and suppose f is continuous at x_0, differentiable near it.*

(1) *Either of the following is sufficient to show that f has a local* minimum *at x_0:*
  (a) $f''(x_0) > 0$ *or;*
  (b) *$f'(x)$ is negative to the left of x_0, positive to its right.*
(2) *Either of the following shows that f has a local* maximum *at x_0:*
  (a) $f''(x_0) < 0$ *or;*
  (b) *$f'(x)$ is positive to the left of x_0, negative to its right.*

Curve sketching protocol. Given a function f.

0th derivative stuff:
  (a) The domain and the domain of continuity.
  (b) Domains where $f > 0$, $f < 0$.
  (c) Anchor points: x- and y-intercepts.
  (d) Horizontal and vertical asymptotes.

1st derivative stuff: using $f'(x)$ determine
  (a) Domains where $f' > 0$, $f' < 0$
  (b) Critical and singular numbers.

2nd derivative stuff:
  (a) Domains where $f'' > 0$, $f'' < 0$
  (b) Points where $f''(x) = 0$, inflection points.

Date: 12/11/2015.

2. Examples

2.1. $f(x) = \frac{x^2-9}{x^2+3}$.

- $f'(x) \overset{\text{quot}}{=} \frac{2x(x^2+3)-2x(x^2-9)}{(x^2+3)^2} = \frac{24x}{(x^2+3)^2}$.
- $f''(x) = 24\frac{1}{(x^2+3)^2} - 24\frac{x\cdot 2\cdot 2x}{(x^2+3)^3} = 24\frac{(x^3+3)-4x^2}{(x^2+3)^3} = 72\frac{1-x^2}{(x^2+3)^3}$.

Thus

(1) f defined on $\mathbb{R}$, cts everywhere (defined by formula; denominator everywhere nonzero). Moreover
   (a) $f(0) = -3$, $f(x) = \frac{(x-3)(x+3)}{x^2+3}$ so vanishes at $x = \pm 3$, negative between them, positive otherwise.
   (b) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1-9/x^2}{1+3/x^2} = 1$.
(2) $f'(x)$ is negative for $x < 0$, zero at $x = 0$, positive at $x > 0$ (hence at the critical number $x = 0$ we have a local minimum)
(3) $f''(x)$ has the same sign as $(1-x)^2 = (1-x)(1+x)$ so it is negative if $x < -1$ or $x > 1$, positive if $-1 < x < +1$ and zero at $x = \pm 1$ which are therefore inflection points.

The "special" points were: $-3, -1, 0, 1, 3$ so we break up the domain of f at those points:

x	$(-\infty, -3)$	-3	$(-3, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, 3)$	3	$(3, \infty)$
f	+	0	-	-2	-	-9	-	-2	-	0	+
f'	+	+	+	+	+	0	+	+	+	+	+
f''	-	-	-	0	-	-	-	0	-	-	-

[Plot to be added]

2.2. $f(x) = x^{2/3}(x-1)$.

- $f'(x) = \frac{2}{3}x^{-1/3}(x-1) + x^{2/3} = \frac{2(x-1)+3x}{3x^{1/3}} = \frac{5x-2}{3x^{1/3}}$
- $f''(x) = \frac{5}{3x^{1/3}} - \frac{5x-2}{9x^{4/3}} = \frac{15x-(5x-2)}{9x^{4/3}} = \frac{10x+2}{9x^{4/3}}$

Thus (note: $x^{2/3}$ and $x^{4/3}$ are always non-negative; $x^{1/3}$ has the same sign as x)

(1) f defined on $\mathbb{R}$ ($x^{1/3}$ defined everywhere), continuous there (defined by formula).
   (a) $f(0) = 0$, $f(1) = 0$ and f is positive if $x < 1$ negative if $x > 1$ ($x^{2/3} \geq 0$ for all x)
   (b) $\lim_{x\to\pm\infty} |f(x)| = \infty$ so no horizontal asymptotes.
(2) The critical numbers are 0 (f' undefined) and $\frac{2}{5}$ ($f' = 0$). Otherwise $f' > 0$ if $x < 0$, $f' < 0$ if $0 < x < \frac{2}{5}$ and $f' > 0$ if $x > \frac{2}{3}$.
(3) . Thus f'' is undefined at 0, vanishes at $-\frac{1}{5}$, and is negztive if $x < -\frac{1}{5}$, positive if $-\frac{1}{5} < x < 0$ or $x > 0$, so only $-\frac{1}{5}$ is an inflection point.

Summary table:

x	$(-\infty, -\frac{1}{5})$	$-\frac{1}{5}$	$(-\frac{1}{5}, 0)$	0	$(0, \frac{2}{5})$	$\frac{2}{5}$	$(\frac{2}{5}, 1)$	1	$(1, \infty)$
f	-	$-\frac{4}{5^{5/3}}$	-	0	-	$-\frac{3\cdot 4^{1/3}}{5^{5.3}}$	-	0	+
f'	+	+	+	undef	-	0	+	+	+
f''	-	0	+	undef	+	+	+	+	+

[Plot to be added]

2.3. **??**