MATH 100 – SOLUTIONS TO WORKSHEET 23
ANTIDERIVATIVES

1. Warmup

(1) (Multiplication)
   (a) Calculate $7 \times 8 = \boxed{15}$
   (b) Find a, b such that $ab = 15$. $\boxed{15 = 1 \times 15 = 3 \times 5 = 5 \times 3 = 15 \times 1}$.

(2) (Trig functions)
   (a) Calculate $\sin \frac{\pi}{3} = \boxed{\dfrac{\sqrt{3}}{2}}$.
   (b) Find all θ such that $\sin \theta = 1$. $\boxed{\theta = \dfrac{\pi}{2} + 2\pi k,\ k \in \mathbb{Z}}$.

(3) Simple differentiation
   (a) Find one f such that $f'(x) = 1$.
   Solution: $\boxed{f(x) = x}$ works.
   (b) Find *all* such f.
   Solution: $\boxed{f(x) = x + c}$, c a constant.
   (c) Find the f such that $f(7) = 3$.
   Solution: Need c such that $3 = f(7) = 7 + c$ so $c = -4$ and $\boxed{f(x) = x - 4}$.

2. Antidifferentiation by massaging

(1) Find f such that $f'(x) = -\frac{1}{x}$.
   Solution: $\frac{\mathrm{d}}{\mathrm{d}x} \log |x| = \frac{1}{x}$ so $\boxed{f(x) = -\log |x|}$ works.

(2) Find f such that $f'(x) = \cos x$.
   Solution: $\boxed{f(x) = \sin x}$ works.

(3) Find all f such that $f'(x) = \cos 3x - \frac{2}{x}$.
   Solution: $(\sin 3x)' = 3 \cos 3x$ so $\boxed{f(x) = \dfrac{1}{3} \sin(3x) - 2 \log |x| + c}$.

(4) Find f such that $f'(x) = 2x^{1/3} - x^{-2/3}$ and $f(1000) = 5$.
   Solution: Since $\left(x^{4/3}\right)' = \frac{4}{3}x^{1/3}$ and $\left(x^{1/3}\right)' = \frac{1}{3}x^{-2/3}$ the general solutions is

$$f(x) = 2 \cdot \frac{3}{4} x^{4/3} - 3x^{1/3} + c\,.$$

To get the specific solution we solve using $(1000)^{1/3} = 10$:

$$\begin{aligned} 5 &= f(1000) = \frac{3}{2}(1000)^{4/3} - 3(1000)^{1/3} + c \\ &= \frac{3}{2} 10^4 - 30 + c \end{aligned}$$

so

$$c = 35 - 15,000 = -14,965$$

and

$$\boxed{f(x) = \frac{3}{2} x^{4/3} - 3x^{1/3} - 14,965\,.}$$

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(5) Find f such that $f''(x) = \sin x + \cos x$, $f(0) = 0$ and $f'(0) = 1$.

Solution: Since $(f')'(x) = \sin x + \cos x$, $f'(x) = -\cos x + \sin x + c$. Now $f'(0) = -1 + 0 + c = 1$ so $c = 2$ and $f'(x) = -\cos x + \sin x + 2$. From this we get $f(x) = -\sin x - \cos x + 2x + d$ for some d. We also need $f(0) = -0 - 1 + 0 + d = 0$ so $d = 1$ and

$$\boxed{f(x) = -\sin x - \cos x + 2x + 1\,.}$$

(6) A cannonball is dropped off a tower of height H. Suppose that it starts from rest at the top of the tower and that its acceleration is constant (equal to g). When does it hit the ground?

Solution: Suppose the height of the cannonball at time t is $y(t)$. We are then given that $a(t) = \ddot{y}(t) = -g$. We first find the velocity. $v(t) = \dot{y}(t)$ satisfies $\dot{v}(t) = a(t) = -g$ so $v(t) = -gt + c$ for a constant c. Since $v(0) = 0$ (starting at rest) we have $c = 0$ and $v(t) = -gt$. This means $\dot{y}(t) = v(t) = -gt$ so $y(t) = -\frac{1}{2}gt^2 + d$ for a constant d. We have $H = y(0) = d$ so $y(t) = -\frac{1}{2}gt^2 + H$.

Endgame: we need to solve for t such that $y(t) = 0$ (hitting the ground), which means

$$\begin{aligned} -\frac{1}{2}gt^2 + H &= 0 \\ \frac{1}{2}gt^2 &= H \\ t^2 &= \frac{2H}{g} \end{aligned}$$

that is

$$\boxed{t = \sqrt{\frac{2H}{g}}\,.}$$