MATH 100 – SOLUTIONS TO WORKSHEET 22
L'HÔPITAL'S RULE

1. Statement

(1) Evaluate $\lim_{x\to1}\frac{\log x}{x-1}$.

Solution 1: By the definition of the derivative this is $\lim_{x\to1}\frac{\log x-\log 1}{x-1}=\frac{\mathrm{d}\log x}{\mathrm{d}x}\restriction_{x=1}=\frac{1}{x}\restriction_{x=1}=\frac{1}{1}=1$.

Solution 2: since $\lim_{x\to1}\log x=\log 1=0$ and $\lim_{x\to1}(x-1)=1-1=0$ this is indeterminate $\frac{0}{0}$; by l'Hôpital's rule,

$$\lim_{x\to1}\frac{\log x}{x-1}=\lim_{x\to1}\frac{1/x}{1}=\frac{1/1}{1}=\boxed{1}.$$

(2) Evaluate $\lim_{x\to0}\frac{\cos x-1}{x^2}$.

Solution: We apply l'Hôpital twice getting:

$$\lim_{x\to0}\frac{\cos x-1}{x^2}=\lim_{x\to0}\frac{-\sin x}{2x}=\lim_{x\to0}\frac{-\cos x}{2}=-\frac{\cos 0}{2}=\boxed{-\frac{1}{2}}.$$

Justification: in the first limit we have since $\lim_{x\to0}(\cos x-1)=\cos 0-1=0$ and $\lim_{x\to0}x^2=0$ so indereminate $\frac{0}{0}$. In the second limit we had $\lim_{x\to0}\sin x=0$ and $\lim_{x\to0}(2x)=0$ so again indeterminate $\frac{0}{0}$.

(3) Do (2) using a 2nd-order Taylor expansion.

Solution 2: Since the MacLaurin expansion of the cosine function is $\cos x=1-\frac{1}{2}x^2+O(x^4)$ (O is the first letter of the word Order, and this says "next term of order at least x^4")

$$\lim_{x\to0}\frac{\cos x-1}{x^2}=\lim_{x\to0}\frac{-\frac{1}{2}x^2+O(x^4)}{x^2}=\lim_{x\to0}\left(-\frac{1}{2}+O(x^2)\right)=-\frac{1}{2}\,.$$

(4) Given that $f(2)=5$, $g(2)=3$, $f'(2)=7$ and $g'(2)=4$ find $\lim_{x\to3}\frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3}$.

Solution: Since f,g are differentiable at 2 they are continuous there. We have $\lim_{x\to3}\left(f(2x-4)-g(x-1)-2\right)=f(2)-g(2)-2=5-3-2=0$ and $\lim_{x\to3}\left(g(x^2-7)-3\right)=g(2)-3=0$ so this is indeterminate $\frac{0}{0}$. By l'Hôpital and the chain rule we have

$$\lim_{x\to3}\frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3}=\lim_{x\to3}\frac{2f'(2x-4)-g'(x-1)}{2x\cdot g'(x^2-7)}=\frac{2f'(2)-g'(2)}{2\cdot3\cdot g'(2)}=\frac{2\cdot7-4}{6\cdot4}=\frac{10}{24}=\boxed{\frac{5}{12}}.$$

(5) Evaluate $\lim_{x\to0^+}\frac{e^x}{x}$.

Solution: As $x\to0$ we have $e^x\to1>0$ while $\frac{1}{x}\to+\infty$ (we have positive x here) so $\lim_{x\to0^+}\frac{e^x}{x}=+\infty$.

Pitfall: Naively applying l'Hôpital seems to give $\lim_{x\to0}\frac{e^x}{x}=\lim_{x\to0}\frac{e^x}{1}=1$ which is wrong because this is not an indeterminate form: the denominator approaches 0 but the numerators approaches 1.

(6) Evaluate $\lim_{x\to\infty}x^2e^{-x}$

Solution: Applying l'Hôpital twice we get

$$\lim_{x\to\infty}x^2e^{-x}=\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0\,,$$

where in each stage this was justified since both e^x and any positive power of x tend to ∞ as $x\to\infty$.

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(7) Evaluate $\lim_{x\to 0^+} x\log x$.

Solution: Write this as $\lim_{x\to 0^+} \frac{\log x}{1/x}$ this is indeterminate of the form $\frac{\infty}{\infty}$ (note that $\lim_{x\to 0^+}\log x = -\infty$). Applying l'Hôpital we get

$$\lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = \boxed{0}\,.$$

Remark: we chose to write $\frac{\log x}{1/x}$ and not $\frac{x}{1/\log x}$ since differentiation would simplify the log (make it a power).

(8) Evaluate $\lim_{x\to\infty} x^n e^{-x}$.

Solution: Applying l'Hôpital n times we get

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = n(n-1)\lim_{x\to\infty} \frac{x^{n-2}}{e^x} = \cdots = n!\lim_{x\to\infty}\frac{1}{e^x} = 0\,.$$

(9) Suppose $a > 0$. Evaluate $\lim_{x\to\infty} x^{-a}\log x$.

Solution: We apply l'Hôpital to the $\frac{\infty}{\infty}$ indereminate form $\frac{\log x}{x^a}$

$$\lim_{x\to\infty} x^{-a}\log x = \lim_{x\to\infty}\frac{\log x}{x^a} = \lim_{x\to\infty}\frac{1/x}{ax^{a-1}} = \lim_{x\to\infty}\frac{1}{ax^a} = 0$$

since $a > 0$.

(10) Evaluate $\lim_{x\to 0}(2x+1)^{1/\sin x}$.

Solution: This is indeterminate 1^∞. Taking logarithms we first compute

$$\begin{aligned}\lim_{x\to 0}\log\left((2x+1)^{1/\sin x}\right) &= \lim_{x\to 0}\frac{1}{\sin x}\log(2x+1) = \lim_{x\to 0}\frac{\log(2x+1)}{\sin x}\\ \text{l'Hôpital} &= \lim_{x\to 0}\frac{2/(2x+1)}{\cos x} = \frac{2/1}{1} = 2\,,\end{aligned}$$

where the use of l'Hôpital was justified since $\lim_{x\to 0}\log(2x+1) = \log 1 = 0$ and $\lim_{x\to 0}\sin x = \sin 0 = 0$. Finally we use the continuity of the exponential function:

$$\begin{aligned}\lim_{x\to 0}(2x+1)^{1/\sin x} &= \lim_{x\to 0}\exp\left\{\log\left((2x+1)^{1/\sin x}\right)\right\}\\ &= \exp\left\{\lim_{x\to 0}\log\left((2x+1)^{1/\sin x}\right)\right\}\\ &= \exp(2) = e^x\,.\end{aligned}$$