MATH 100 – SOLUTIONS TO WORKSHEET 16
MINIMA AND MAXIMA

1. Absolute minima and maxima by hand

(1) Find the absolute maximum and minimum values of $f(x) = |x|$ on the interval $[-3, 5]$.
Solution: Absolute minimum 0 attained at $x = 0$, absolute maximum 5 attained at $x = 5$.

(2) Find the absolute maximum and minimum of $f(x) = \sqrt{x}$ on $[0, 5]$.
Solution: This function is *strictly increasing* so its absolute minimum is $f(0) = 0$ and its absolute maximum is $f(5) = \sqrt{5}$.

2. Local minima and derivatives

(1) (Final, 2011) Let $f(x) = 6x^{1/5} + x^{6/5}$.

(a) Find the critical numbers and singularities of f.
Solution: f is defined on $\mathbb{R}$ (note the odd denominator in the power) and continuous there (defined by formula). We have

$$f'(x) = 6 \cdot \frac{1}{5}x^{-4/5} + \frac{6}{5}x^{1/5} = \frac{6}{5}\left(x^{-4/5} + x^{1/5}\right)$$

which exists for all $x \neq 0$ so $\boxed{c = 0}$ is a singular point. For critical points we need to solve $f'(c) = 0$ that is

$$\begin{aligned} \frac{6}{5}\left(\frac{1}{c^{4/5}} + c^{1/5}\right) &= 0 \\ 1 + c &= 0 \\ c &= -1 \end{aligned}$$

and we have a critical point at $\boxed{c = 1}$.

(b) Find its absolute maximum and minimum on the interval $[-32, 32]$.
Solution: Since f is continuous it's enough to check the critical points, singular points and endpoints of the internal. We have: $f(-32) = 6(-2)+(-2)^6 = 52$, $f(-1) = 6(-1)+(-1)^6 = -5$ and $f(32) = 6(2)+2^6 = 76$, so the absolute minimum is -5 attained at $x = -1$ and the absolute maximum is 76 attained at $x = 32$.

(c)

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(2) (caution)

(a) Show that $f(x) = (x-1)^4 + 7$ attains its absolute minimum at $x = 1$.
Solution: $(x-1)^4$ is a square so its absolute minimum value is zero, attained when $x = 1$.

(b) Show that $f(x) = (x-1)^3 + 7$ has $f'(1) = 0$ but has no local minimum or maximum there.
Solution: $(x-1)^3$ is an odd power, so is strictly increasing and has no minima or maxima. Nevertheless $f'(x) = 3(x-1)^2$ and $f'(1) = 0$.

(3) (Midterm, 2010) Find the maximum value of $x\sqrt{1 - \frac{3}{4}x^2}$ on the interval $[0, 1]$.

Solution: Let $f(x) = x\sqrt{1 - \frac{3}{4}x^2}$. Then

$$\begin{aligned} f'(x) &= \sqrt{1 - \frac{3}{4}x^2} + \frac{1}{2}x\frac{-\frac{3}{4}\cdot 2x}{\sqrt{1-\frac{3}{4}x^2}} = \sqrt{1 - \frac{3}{4}x^2} - \frac{3}{4}\frac{x^2}{\sqrt{1-\frac{3}{4}x^2}} \\ &= \frac{\left(1 - \frac{3}{4}x^2\right) - \frac{3}{4}x^2}{\sqrt{1-\frac{3}{4}x^2}} = \frac{1 - \frac{3}{2}x^2}{\sqrt{1-\frac{3}{4}x^2}}\,. \end{aligned}$$

This is defined for $|x| < \frac{2}{\sqrt{3}}$ and in particular in $[0, 1]$ so f' is defined in the whole interval and f is continuous there and has so singular points. It has critical points where $f'(c) = 0$ that is when

$$\frac{1 - \frac{3}{2}c^2}{\sqrt{1-\frac{3}{4}c^2}} = 0\,.$$

Clearing the denominator we get

$$\begin{aligned} 1 - \frac{3}{2}c^2 &= 0 \\ c^2 &= \frac{2}{3} \\ c &= \sqrt{\frac{2}{3}} \end{aligned}$$

so only one criticial point, $c = \sqrt{\frac{2}{3}}$ (note that the other root is outside the interval $[0, 1]$). To find the maximum value we then need to consider $f(0) = 0$, $f(1) = 1\sqrt{1 - \frac{3}{4}} = \frac{1}{2}$ and $f\left(\sqrt{\frac{2}{3}}\right)$ which is

$$f\left(\sqrt{\frac{2}{3}}\right) = \sqrt{\frac{2}{3}}\sqrt{1 - \frac{3}{4}\cdot\frac{2}{3}} = \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} = \frac{1}{2}\,.$$

It follows that the absolute maximum is $\boxed{\frac{1}{\sqrt{3}}}$, attained at $x = \sqrt{\frac{2}{3}}$.

(4) (Final, 2007) Let $f(x) = x\sqrt{3-x}$.

(a) Find the domain of f.
Solution: f is defined where the number under the root is non-negative, that is when $3 - x \geq 0$ that is when $x \leq 3$.

(b) Determine the x-coordinates of any local maxima or minima of f.
Solution: Differentiating we have

$$f'(x) = \sqrt{3-x} - \frac{x}{2\sqrt{3-x}} = \frac{3-x}{\sqrt{3-x}} - \frac{x/2}{\sqrt{3-x}} = \frac{3 - \frac{3}{2}x}{\sqrt{3-x}} = \frac{3}{2}\frac{2-x}{\sqrt{3-x}}\,.$$

and we see that $f'(x)$ is defined for all $x < 3$ so any local minimum or maximum will satisfy $f'(c) = 0$. We see that this can only happen when the numerator vanishes, that is at $c = 2$. Since for $c < 2$ the derivative is positive (f increasing) while for $2 < c < 3$ the derivative is negative (f decreasing) we see that the point is a *local maximum.* Since $f(c) = 2\sqrt{3-2} = 2$ the *coordinates* of the unique local maximum are $\boxed{(2, 2)}$ and there are *no local minima.*