MATH 100 – SOLUTIONS TO WORKSHEET 15
ESTIMATES ON TAYLOR APPROXIMATIONS

1. Taylor approximations

(1) Find the 1^{st} and 2^{nd} order Taylor expansions of $x^{3/2}$ about $x = 4$ and use them to approximate $(4.1)^{3/2}$.

Solution: let $f(x) = x^{3/2}$ and work about $a = 4$. Then $f'(x) = \frac{3}{2}x^{1/2}$, $f''(x) = \frac{3}{4}x^{-1/2}$, so $f(4) = 2^3 = 8$, $f'(4) = \frac{3}{2} \cdot 2 = 3$ and $f''(4) = \frac{3}{4}2^{-1} = \frac{3}{8}$.

$$\begin{aligned} T_1(x) &= f(a) + f'(a)(x-a) = 8 + 3(x-4) \\ T_2(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 = 8 + 3(x-4) + \frac{3}{16}(x-4)^2 . \end{aligned}$$

Plugging in $x = 4.1$, $x - a = \frac{1}{10}$ the linear approximation is $\boxed{(4.1)^{3/2} \approx 8 + \frac{3}{10} = 8.3}$ and the quadratic approximation is $\boxed{(4.1)^{3/2} \approx 8 + \frac{3}{10} + \frac{3}{1600}}$.

(2) Find the 2^{nd} order Taylor expansion of $x^{3/2} + 3x$ about $x = 4$.

See solution to worksheet 14

(3) Find the 8th order expansion of $f(x) = e^{x^2} + \cos(5x)$. What is $f^{(6)}(0)$?

See solution to worksheet 14

2. Error estimates

> Let $R_1(x) = f(x) - T_1(x)$ be the *remainder*. Then there is c between a and x such that
> $$R_1(x) = \frac{f^{(2)}(c)}{2!}(x-a)^2$$

(1) Estimate the error in the linear approximation to $(4.1)^{3/2}$.

Solution: Continuing with $f(x) = x^{3/2}$, by the Lagrange form of the remainder, we have

$$R_1(x) = \frac{1}{2}f^{(2)}(c)(x-a)^2$$

for some c between x and a. For us $x = 4.1$, $a = 4$ so there is $c \in (4, 4.1)$ such that

$$R_1(4.1) = \frac{1}{2}\left(\frac{3}{8}c^{-1/2}\right)(4.1-4)^2 = \frac{3}{1600}c^{-1/2} .$$

Now the function $c^{-1/2}$ is *decreasing* on $[4, 4.1]$ so $c^{-1/2} \le 4^{-1/2} = \frac{1}{2}$ and

$$\boxed{R_1(4.1) \le \frac{3}{3200} < \frac{1}{1000}} .$$

> Let $R_n(x) = f(x) - T_n(x)$ be the *remainder*. Then there is c between a and x such that
> $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

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(2) Estimate the error in the quadratic approximation to $(4.1)^{3/2}$.

Solution: By the Lagrange form of the remainder, we have

$$R_2(x) = \frac{1}{3!} f^{(3)}(c)(x-a)^3$$

for some c between x and a. Since $f^{(2)}(x) = \frac{3}{8}x^{-1/2}$ we get $f^{(3)}(x) = -\frac{3}{16}x^{-3/2}$. For us $x = 4.1$, $a = 4$ so there is $c \in (4, 4.1)$ such that

$$R_2(4.1) = \frac{1}{6}\left(\frac{3}{16}c^{-3/2}\right)(4.1-4)^3 = -\frac{1}{32,000}c^{-3/2}\,.$$

This is negative, so we have an over-estimate. Also, the function $c^{-3/2}$ is *decreasing* on $[4, 4.1]$ so $c^{-3/2} \leq 4^{-3/2} = \frac{1}{8}$ and

$$\boxed{|R_2(4.1)| \leq \frac{1}{32,000} \cdot \frac{1}{8} = \frac{1}{256,000}\,.}$$

(3) Estimate the error in the 4th order approximation to $\cos(0.5)$

Solution: Expanding $f(x) = \cos(x)$ about zero, by the Lagrange form of the remainder, we have $0 < c < 0.5$ for which

$$R_4(x) = \frac{f^{(5)}(c)}{5!}(x-a)^5$$

that is

$$R_4(0.5) = \frac{-\sin(c)}{5!}(0.5)^5 = -\frac{\sin(c)}{32 \cdot 120}\,.$$

Now $|\sin(c)| \leq 1$ so

$$\boxed{|R_4(0.5)| \leq \frac{1}{32 \cdot 120} = \frac{1}{3840}\,.}$$

Common error: The 4^{th} order expansion reads $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$. There is no x^5 term since $f^{(5)}(a) = f^{(5)}(0) = 0$. But THIS DOES NOT MEAN that $f^{(5)}(c) = 0$ – the value of c for which the Lagrange form holds need not be an endpoint (in fact, it never is).