MATH 100 – SOLUTION TO WORKSHEET 10
LOGARITHMS AND THEIR DERIVATIVES

1. Inverse Trig & Differentiation

(1) The angle θ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and satisfies $\sin(\theta) = 0.4$. find $\tan\theta$.

Solution: By Pythagoras we have $\sin^2\theta + \cos^2\theta = 1$ so $\cos^2\theta = 1 - 0.4^2 = 0.84$. In the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ the cosine function is positive, so $\cos\theta = \sqrt{0.84}$ and

$$\boxed{\tan\theta = \frac{0.4}{\sqrt{0.84}}.}$$

(2) (Final 2011) Find the derivative of $\arcsin(3x+1)$

Solution: Applying the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(3x+1) = \frac{1}{\sqrt{1-(3x+1)^2}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}[3x+1] = \frac{3}{\sqrt{-6x-9x^2}}.$$

2. Review of Logarithms

(1) $\log\left(e^{10}\right) = \boxed{10}$ $\qquad$ $\log(2^{100}) = 100\log 2$

(2) A variant on *Moore's Law* states that computing power doubles every 18 months.
Suppose computers today can do N_0 operations per second.

(a) Write a formula for the power of computers t years into the future:

- Computers t years from now will be able to do $N(t)$ operations per second where

$$N(t) = N_0 2^{t/1.5}$$

Explanantion: we are given that there is a doubling every 18 months, so $t/1.5$ doublings in t years.

(b) A computing task would take 10 years for today's computers. Suppose we wait 3 years and then start the computation. When will we have the answer?

Solution: In 3 years we will see 2 doublings, so the computers will be $2^2 = 4$ times as powerful and will complete the computation in $\frac{10}{4} = 2.5$ years. We'll then have the answer $\boxed{\text{5.5 years from now}}$ (don't forget the endgame!)

(c) At what time will computers be powerful enough to complete the task in 6 months?

Solution: t years from now computers will complete the task in $\frac{10}{2^{t/1.5}}$ years, so we need to find t such that

$$\frac{10}{2^{t/1.5}} = \frac{1}{2}.$$

Clearing denominators this gives

$$20 = 2^{t/1.5}.$$

Taking logarithms we get

$$\log 20 = \frac{t}{1.5}\log 2$$

so

$$t = 1.5\frac{\log 20}{\log 2} \text{ years.}$$

Date: 13/10/2015, Worksheet by Lior Silberman.

(1) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} = \frac{1}{ax} \cdot a = \boxed{\frac{1}{x}}$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2+3t\right) = \frac{1}{t^2+3t} \cdot (2t+3) = \boxed{\frac{2t+3}{t^2+3t}}$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1+x^2) \overset{\text{pdt}}{=} 2x\log(1+x^2) + x^2 \frac{\mathrm{d}}{\mathrm{d}x} \log\left(1+x^2\right) \overset{\text{chain}}{=} 2x\log(1+x^2) + x^2 \frac{1}{1+x^2} \cdot 2x$ so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 \log(1+x^2)\right] = 2x\log(1+x^2) + \frac{2x^3}{1+x^2}} .$$

$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} \overset{\text{chain}}{=} -\frac{1}{\log^2(2+\sin r)} \frac{\mathrm{d}}{\mathrm{d}r}\left[\log(2+\sin r)\right] \overset{\text{chain}}{=} -\frac{1}{\log^2(2+\sin r)} \frac{1}{2+\sin r} \frac{\mathrm{d}}{\mathrm{d}r}\left[2+\sin r\right]$ so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} = -\frac{\cos r}{(2+\sin r)\log^2(2+\sin r)}} .$$

(c) Find y' if $\log(x+y) = e^y$.
Solution: We differentiate both sides to get:

$$\frac{1}{x+y}(1+y') = e^y y' .$$

We now solve for y':

$$\begin{aligned} \frac{1}{x+y} + \frac{y'}{x+y} &= e^y y' \\ \frac{1}{x+y} &= \left(e^y - \frac{1}{x+y}\right) y' \\ y' &= \frac{1/(x+y)}{e^y - 1/(x+y)} = \frac{1}{(x+y)e^y - 1} . \end{aligned}$$