Math 322: Problem Set 9 (due 13/11/2014)

P1. In class we classified the groups of order 12, finding the isomorphism types A_{12}, C_{12}, $C_4 \ltimes C_3$, $C_2 \times C_6$, $C_2 \ltimes S_6$. The dihedral group D_{12} is a group of order 12 – where does it fall in this classification?

Sylow's Theorems

1. Let G be a group of order n, and for each $p|n$ let P_p be a p-Sylow subgroup.
   (a) Show that $\left\langle \cup_{p|n} P_p \right\rangle = G$.
   (*b) Suppose that G has a unique p-Sylow subgroup for each p. Show that $G \simeq \prod_{p|n} P_p$ (direct product).

2. Show that there is no simple group of order 36 (hint: construct a non-trivial action on a set of size 4).

**3. G be a finite group, $P < G$ a Sylow subgroup. Show that $N_G(N_G(P)) = N_G(P)$ (hint: let $g \in N_G(N_G(P))$ and consider the subgroup gPg^{-1}).

4. Let G be a group of order $255 = 3 \cdot 5 \cdot 17$.
   (a) Show that $n_{17}(G) = 1$.
   (*b) Show that P_{17} is central in G (hint: conjugation gives a homomorphism $G \to \mathrm{Aut}(P_{17})$).
   (*c) Show that $n_5(G) = 1$
   (d) Show that P_5 is also central in G.
   (e) Show that $G \simeq C_3 \times C_5 \times C_{17} \simeq C_{255}$.

5. Let G be a group of order 140
   (a) Show that $G \simeq H \ltimes C_{35}$ where H is a group of order 4.
   (*b) Classify actions of C_4 on C_{35} and determine the isomorphism classes of groups of order 140 with $P_2 \simeq C_4$.
   (**c) Classify actions of V on C_{35} and determine the isomorphism classes of groups of order 140 with $P_2 \simeq V$.