MATH 100 – SOLVED WORKSHEET 4
CONTINUITY, HORIZONTAL ASYMPTOTES, THE DERIVATIVE

1. The Intermediate Value Theorem

(1) Show that:

(a) $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \le x \le 1$.
Solution: f is continuous on $[0,1]$ (polynomial) and $f(0) = 1$, $f(1) = -2$. By the IVT, f takes the value 0 which lies between $1, -2$.

(b) There is $x > 0$ for which $\frac{1}{x} = \sin x$.
Solution 1: Let $h(x) = \frac{1}{x} - \sin x$. Then h is continuous for $x > 0$ (defined by formula). $h\left(\frac{1}{100}\right) = 100 - \sin\frac{1}{100} \ge 100 - 1 = 99 > 0$ while $h\left(\frac{\pi}{2}\right) = \frac{2}{\pi} - \sin\frac{\pi}{2} = \frac{2}{\pi} - 1 = \frac{2-\pi}{\pi} < 0$ since $\pi > 3$. Conclusion: $h(x)$ changes sign between $\frac{1}{100}$ and $\frac{\pi}{2}$, so $h(x) = 0$ somwhere in between, at which point $\frac{1}{x} = \sin x$.
Solution 2: The functions $\frac{1}{x}, \sin x$ are continuous for $x > 0$. At $a = \frac{1}{100}$ we have $\sin\frac{1}{100} \le 1 < 100 = \frac{1}{1/100}$. At $b = \frac{\pi}{2}$ we have $\sin\frac{\pi}{2} = 1 > \frac{2}{\pi}$ so $\frac{1}{a} > \sin a$ while $\frac{1}{b} < \sin b$ so for some x between a, b we have $\frac{1}{x} = \sin x$.

(2) (Final 2011) Let $y = f(x)$ be continuous with domain $[0,1]$ and range in $[3,5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.

Solution: Let $h(x) = f(x) - (2x+3)$. If $h(x) = 0$ then $f(x) = 2x + 3$ and we will be done. $h(x)$ is continuous on $[0,1]$ (formula using continuous functions). $h(0) = f(0) - 3$ is in the range $[0,2]$ because $f(0)$ is in the range $[3,5]$. $h(1) = f(1) - 5$ is in the range $[-2,0]$ because $f(1)$ is in the range $[3,5]$. If $h(0) = 0$ or $h(1) = 0$ then $x = 0$ or $x = 1$ would work. Otherwise, $h(0) > 0$ (recall the range) and $h(1) < 0$ so by the IVT, $h(x) = 0$ for some x.

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2. Horizontal Asymptotes

(1) Evaluate the following limits:

(a) $\lim_{x\to\infty}\frac{x^2+1}{x-3}=\lim_{x\to\infty}\frac{x+1/x}{1-3/x}=\frac{\lim_{x\to\infty}\left(x+\frac{1}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}=\frac{\infty}{1}=\infty.$

(b) $\lim_{x\to\infty}\frac{x^2+8}{2x^3-1}=\lim_{x\to\infty}\frac{1/x+8/x^3}{2-1/x^3}=\frac{\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{8}{x^3}}{2-\lim_{x\to\infty}\frac{1}{x^3}}=\frac{0+0}{2-0}=0.$

(c) $\lim_{x\to\infty}\frac{\sqrt{x^4+\sin x}}{x^2-\cos x}=\lim_{x\to\infty}\frac{\sqrt{x^4+\sin x}}{x^2-\cos x}\cdot\frac{x^2}{\sqrt{x^4}}=\lim_{x\to\infty}\frac{\sqrt{\frac{x^4+\sin x}{x^4}}}{\frac{x^2-\cos x}{x^2}}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{\sin x}{x^4}}}{1-\frac{\cos x}{x^2}}.$

Side calculation: For all x, $-1\le\sin x\le 1$ so $-\frac{1}{x^4}\le\frac{\sin x}{x^4}\le\frac{1}{x^4}$. Since $\lim_{x\to\infty}\frac{1}{x^4}=0=\lim_{x\to\infty}-\frac{1}{x^4}$, by the squeeze theorem also $\lim_{x\to\infty}\frac{\sin x}{x^4}=0$. Similarly, for all x $-1\le\cos x\le 1$ so $-\frac{1}{x^2}\le\frac{\cos x}{x^2}\le\frac{1}{x^2}$. Since $\lim_{x\to\infty}\pm\frac{1}{x^2}=\pm\lim_{x\to\infty}\frac{1}{x^2}=0$, we have $\lim_{x\to\infty}\frac{\cos x}{x^2}=0$.

Conclusion: Applying the limit laws we get

$$\lim_{x\to\infty}\frac{\sqrt{x^4+\sin x}}{x^2-\cos x}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{\sin x}{x^4}}}{1-\frac{\cos x}{x^2}}=\frac{\sqrt{1+\lim_{x\to\infty}\frac{\sin x}{x^4}}}{1-\lim_{x\to\infty}\frac{\cos x}{x^2}}=\frac{\sqrt{1+0}}{1-0}=1$$

(d) $\lim_{x\to-\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-1}\right)$

Solution 1: We have

$$\begin{aligned}\sqrt{x^2+2x}-\sqrt{x^2-1} &= \frac{\left(\sqrt{x^2+2x}-\sqrt{x^2-1}\right)\left(\sqrt{x^2+2x}+\sqrt{x^2-1}\right)}{\sqrt{x^2+2x}+\sqrt{x^2-1}}\\ &= \frac{\left(x^2+2x\right)-\left(x^2-1\right)}{\sqrt{x^2+2x}+\sqrt{x^2-1}}=\frac{2x+1}{\sqrt{x^2+2x}+\sqrt{x^2-1}}.\end{aligned}$$

[aside: the numerator and the denominator are roughly of order x] divide the fraction by $1=\frac{-x}{\sqrt{x^2}}$ (since $x\to-\infty$, x is eventually negative so $\sqrt{x^2}=-x$) go get:

$$\begin{aligned}\sqrt{x^2+2x}-\sqrt{x^2-1} &= \frac{2x+1}{\sqrt{x^2+2x}+\sqrt{x^2-1}}\Big/\frac{-x}{\sqrt{x^2}}\\ &= \frac{\frac{2x+1}{-x}}{\sqrt{\frac{x^2+2x}{x^2}}+\sqrt{\frac{x^2-1}{x^2}}}=\frac{-2-\frac{1}{x}}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x^2}}}.\end{aligned}$$

We can now use the limit laws:

$$\begin{aligned}\lim_{x\to-\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-1}\right) &= \lim_{x\to-\infty}\frac{-2-\frac{1}{x}}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x^2}}}\\ &= \frac{-\lim_{x\to-\infty}2-\lim_{x\to-\infty}\frac{1}{x}}{\sqrt{1+\lim_{x\to-\infty}\frac{2}{x}}+\sqrt{1-\lim_{x\to-\infty}\frac{1}{x^2}}}\\ &= \frac{-2-0}{\sqrt{1+0}+\sqrt{1-0}}=\frac{-2}{2}=-1.\end{aligned}$$

Solution 2: As above, we need to evaluate

$$\lim_{x\to-\infty}\frac{2x+1}{\sqrt{x^2+2x}+\sqrt{x^2-1}}.$$

Now change variables via $x=-u$ so that $u\to\infty$. The expression becomes:

$$\lim_{u\to\infty}\frac{-2u+1}{\sqrt{u^2-2u}+\sqrt{u^2-1}}=\lim_{u\to\infty}\frac{-2+\frac{1}{u}}{\sqrt{1-\frac{2}{u}}+\sqrt{1-\frac{1}{u^2}}}=\frac{-2+0}{\sqrt{1-0}+\sqrt{1-0}}=-1.$$